A. Lengths of curves

Suppose that $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is a parametrization of a curve $C$ stretching from $\mathbf{x}(a)$ to $\mathbf{x}(b)$, and that $\mathbf{x}(t)$ has smooth component functions. If $\mathbf{x}$ represents the trajectory of a moving particle and $\Delta t$ is a small increment in $t$, the distance traversed by the particle in the time interval $\Delta t$ is approximately

$$\Delta s = |\mathbf{x}(t + \Delta t) - \mathbf{x}(t)| = \left| \frac{\mathbf{x}(t + \Delta t) - \mathbf{x}(t)}{\Delta t} \right| \Delta t.$$ 

In the limit as $\Delta t \to 0$, we obtain an equality of differentials,

$$ds = \left| \frac{d\mathbf{x}}{dt} \right| dt. \quad (1)$$

Here $\frac{d\mathbf{x}}{dt}$ is the derivative of the vector-valued function $\mathbf{x}(t)$ with respect to $t$, the differentiation being performed componentwise:

$$\frac{d\mathbf{x}}{dt} = \left( \frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt} \right).$$

We can think of equation (1) as stating that the distance traversed by a moving particle in a small time interval is the speed of the particle times the length of the time interval. To find the length of the parametrized curve $\mathbf{x} : [a, b] \to \mathbb{R}^n$, we simply integrate (1) from $t = a$ to $t = b$:

$$\text{Length of the curve } C = \int_a^b \left| \frac{d\mathbf{x}}{dt} \right| dt.$$ 

We can write this out in terms of components as:

$$\text{Length of the curve } C = \int_a^b \sqrt{(dx_1/dt)^2 + \ldots + (dx_n/dt)^2} dt.$$
Problem 6.1. Find the length of the parametrized helix

\[ \mathbf{x} : [0, 2\pi] \to \mathbb{R}^3 \text{ defined by } \mathbf{x}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}. \]

The chain rule can be used to give a convenient formula for calculating lengths of curves which are expressed in polar coordinates. Recall that polar coordinates \( r, \theta \) are related to the usual coordinates \( x, y \) by the equations

\[ x = r \cos \theta, \quad y = r \sin \theta. \]

If \( r \) and \( \theta \) are functions of \( t \), then

\[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 = \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\theta}{dt} \right)^2, \]

which implies the following formula for arc length in polar coordinates:

\[ \text{Length of the curve } \mathbf{C} = \int_a^b \sqrt{\left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\theta}{dt} \right)^2} \, dt. \]

Problem 6.2. Find the length of the spiral, defined in polar coordinates by the formula,

\[ r = e^\theta, \quad 0 \leq \theta \leq 2\pi. \]

Digression. Special relativity and the twin paradox

Special relativity provides a striking application of a slightly altered arc length integral. Here are two quotations from the creators of the theory:

I am convinced that the philosophers have had a harmful effect upon the progress of scientific thinking in removing certain fundamental concepts from the domain of empiricism, where they are under our control, to the intangible heights of the \textit{a priori}. For even if it should appear that the universe of ideas cannot be deduced from experience by logical means, but is, in a sense, a creation of the human mind, without which no science is possible, nevertheless this universe of ideas is just as little independent of the nature of our experiences as clothes are of the form of the human body. This is particularly true of our concepts of time and space, which physicists have been obliged by the facts to bring down from the Olympus of the \textit{a priori} in order to adjust them and put them in a serviceable condition. (Albert Einstein, \textit{The meaning of relativity}, Princeton Univ. Press, Princeton, New Jersey, 1955, p.2.)
Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality. (Hermann Minkowski, *Space and time*, in *The principle of relativity*, Dover, New York, 1923, p.75.)

Special relativity teaches that the time measured by a clock depends upon its motion. The arena for special relativity is space-time, which is given space coordinates \((x, y, z)\) and a time coordinate \(t\). One imagines that the points of space-time locate *events* which occur at a specific location and at a specific time—for example, the exploding of a firecracker at a given location \((x, y, z)\) and at a given time \(t\) would be regarded as an event.

An individual living within space-time witnesses a continuum of events, represented by a curve in space-time called his world line. The curve starts at an event, the birth of the individual, located at a point \((t_0, x_0, y_0, z_0)\), and ends at another event, the death of the individual, located at a point \((t_1, x_1, y_1, z_1)\). Such a curve is called a *world line*.

One of the tenets of special relativity is that nothing can move faster than the speed of light. Thus, if \(c\) denotes the speed of light,

\[
\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \leq c.
\]

If we square both sides, and multiply by \(dt^2\), this inequality becomes

\[
c^2dt^2 - dx^2 - dy^2 - dz^2 \geq 0.
\]

The subjective time measured by the individual is measured by the “element of arc length,”

\[
ds^2 = dt^2 - \frac{dx^2}{c^2} - \frac{dy^2}{c^2} - \frac{dz^2}{c^2} = \frac{c^2dt^2 - dx^2 - dy^2 - dz^2}{c^2}.
\]

Suppose that \(C\) is an individual’s world line, and let us parametrize it with a parameter \(\tau\), say

\[
x(\tau) = \begin{pmatrix} t(\tau) \\
x(\tau) \\
y(\tau) \\
z(\tau) \end{pmatrix}, \quad a \leq \tau \leq b.
\]

Then the time measured by the individual between the two events \(x(\tau_0)\) and \(x(\tau_1)\) is given by

\[
\int_C ds = \int_C \frac{1}{c} \sqrt{c^2 dt^2 - dx^2 - dy^2 - dz^2}
\]

\[
= \left. \int_{\tau_0}^{\tau_1} \frac{1}{c} \sqrt{c^2 (dt/d\tau)^2 - (dx/d\tau)^2 - (dy/d\tau)^2 - (dz/d\tau)^2} d\tau. \right.
\]
Just like in the case of arc length, this integral depends only on the world line $C$, not on the choice of parameter $\tau$.

Optional Problem 6.3. If an observer is not moving, $\frac{dx}{d\tau} = \frac{dy}{d\tau} = \frac{dz}{d\tau} = 0$. Show that in this case, the time measured by a stationary clock is given by the $t$-coordinate.

Optional Problem 6.4. Suppose that $C = C_1 + C_2$, where $C_1$ is the straight line segment from from the event $(0, 0, 0, 0)$ to the event $(T, 0, 0, .99cT)$ and $C_2$ is the straight line segment from from the event $(T, 0, 0, .99cT)$ to the event $(2T, 0, 0, 0)$. Find the time that is measured by an observer that moves along the world line $C$. Show that this is smaller than the time measure by an observer which moves along the straight line segment from $(0, 0, 0, 0)$ to $(2T, 0, 0, 0)$.

This is known as the twin paradox.

B. Integrating functions along curves

A smooth curve $C$ is said to be regular if it possesses a smooth parametrization $x : \mathbb{R} \to \mathbb{R}^n$ or $x : [a, b] \to \mathbb{R}^n$ whose velocity $x'(t)$ is never zero. A parametrization $x : \mathbb{R} \to \mathbb{R}^n$ or $x : [a, b] \to \mathbb{R}^n$ is said to be of unit speed, or a parametrization by arc length if $|x'(t)| = 1$, for all $t$.

Theorem. Any regular curve in $\mathbb{R}^n$ possesses a unit-speed parametrization. This parametrization is unique up to the addition of a constant to the time parameter, and a possible change of sign.

This theorem, proven in advanced texts, is very important from a theoretical viewpoint, but it is often quite difficult to actually find the explicit arc length parametrization. A case where one can find the unit speed parametrization is the catenary, the curve in $\mathbb{R}^2$ defined by the equation

$$y = \cosh x = \frac{1}{2}(e^x + e^{-x}).$$

We can parametrize this curve by

$$x : \mathbb{R} \to \mathbb{R}^2, \quad \text{where} \quad x(t) = \left( \begin{array}{c} t \\ \cosh t \end{array} \right),$$

but this is not a unit speed parametrization.

Problem 6.5. a. To find the unit-speed parametrization, we use the formula for Euclidean length,

$$ds^2 = dx^2 + dy^2.$$ 

Using this formula, show that

$$ds = \cosh t \, dt.$$

b. Integration yields

$$s = \sinh t + c,$$
where $c$ is a constant. Take the constant to be zero, so that

$$s = \sinh t = \frac{1}{2}(e^t + e^{-t}), \quad \text{or} \quad (e^t)^2 - 2se^t - 1 = 0.$$  

Solve for $t$ as a function of $s$, $t = \phi(s)$.

c. Use the fact that $s = \sinh t$ to show that $\cosh t = \sqrt{s^2 + 1}$.

d. You can now use $s$ as a parameter on the curve by replacing $x(t)$ by

$$\tilde{x}(s) = x(\phi(s)).$$

Write out the resulting unit-speed parametrization of the catenary.

**Problem 6.6.** Find a unit speed parametrization of the line segment from $(0, 0)$ to $(3, 4)$.

**Problem 6.7.** Find a unit speed parametrization of the circle $x^2 + y^2 = 4$.

If $C$ is a smooth curve in $(x_1, \ldots, x_n)$-space with unit-speed parametrization

$$x : [a, b] \rightarrow \mathbb{R}^n,$$

and $f(x_1, \ldots, x_n)$ a continuous function of two variables, the line integral of $f$ along $C$ is

$$\int_C f(x_1, \ldots, x_n)ds = \int_a^b f(x_1(s), \ldots, x_n(s))ds.$$

In the case where $n = 2$ and $f(x, y) > 0$, this integral has a simple geometric interpretation:

$$\int_C f(x, y)ds = \text{Area of } S,$$

where

$$S = \{(x, y, z) : (x, y) \text{ lies on } C, 0 \leq z \leq f(x, y)\}.$$  

Since it is often difficult to find the unit speed parametrization, it is very important to be able to calculate line integrals without first parametrizing curves with a unit speed parameter. Fortunately, it is possible to use an arbitrary parameter, if we replace everyting in the integrand, including the element of arc length $ds$, by its expression in terms of this parameter. Since

$$ds = \sqrt{(dx_1/\text{dt})^2 + \ldots + (dx_n/\text{dt}^2)\text{dt}},$$

we obtain the following formula for the line integral along a smooth curve $C$ in terms of an arbitrary parametrization $x : [a, b] \rightarrow \mathbb{R}^n$:

$$\int_C f(x_1, \ldots, x_n)ds = \int_a^b f(x_1(t), \ldots, x_n(t))\sqrt{(dx_1/\text{dt})^2 + \ldots + (dx_n/\text{dt}^2)\text{dt}}\text{dt}.$$  

**Problem 6.8.** Calculate the line integral

$$\int_C xds,$$
where $C$ is the part of the parabola in the $(x,y)$-plane defined by 
\[ y = x^2, \quad 0 \leq x \leq 4. \]

Hint: You can use the parametrization
\[ x : [0, 4] \rightarrow \mathbb{R}^2 \quad \text{defined by} \quad x(t) = \left( \frac{t}{t^2} \right). \]

Line integrals can be used to calculate the average value of a function $f(x, y)$ along a curve $C$ in the $(x, y)$-plane, by means of the formula
\[ \text{Average value of } f = \frac{\int_C f(x, y) \, ds}{\text{Length of } C}. \]

Problem 6.9. Find the center of mass of a uniform semicircular wire. The center of mass of a wire of uniform density bent in the form of a curve $C$ is the point whose coordinates are given by
\[ \bar{x} = \text{Average value of } x \text{ on } C = \frac{\int_C x \, ds}{\text{Length of } C}, \]
\[ \bar{y} = \text{Average value of } y \text{ on } C = \frac{\int_C y \, ds}{\text{Length of } C}. \]

For semicircle of radius $a$, we can use the parametrization
\[ x : [0, \pi] \rightarrow \mathbb{R}^2 \quad \text{defined by} \quad x(t) = \left( \frac{a \cos t}{a \sin t} \right). \]

a. Show that $ds = adt$.
b. Determine the line integral \[ \int_C y \, ds. \]
c. Determine the coordinates of the center of mass of the semicircle.

Problem 6.10. Find the line integral
\[ \int_C f(x, y) \, ds, \]
where $f(x, y) = x + 3y + 1$ and $C$ is the straight line segment in $\mathbb{R}^2$ from $(0, 0)$ to $(3, 4)$.

C. Line integrals and work

Suppose that $C$ is a directed regular curve in the plane, that is, a curve with a sense of direction. Let $x : [a, b] \rightarrow \mathbb{R}^2$ is a parametrization such that $dx/dt$
is never zero and as $t$ increases, $C$ is traversed in the positive direction. The orientation picks out a unit tangent vector to $C$, the unit-length vector

$$T(t) = \frac{x'(t)}{|x'(t)|}.$$ 

Given a smooth vector field

$$F(x, y) = M(x, y)i + N(x, y)j,$$

we can form the line integral of the vector field $F$ along $C$,

$$\int_C F \cdot T ds,$$

where $s$ is the arc length parameter that you studied in the previous section. There are some very useful alternate notations for this line integral. Since

$$Tds = T \frac{ds}{dt} dt = x'(t)dt = dx = dx + dy,$$

we can write

$$\int_C F \cdot T ds = \int_C F \cdot dx = \int_C M dx + N dy.$$

An important interpretation of this line integral occurs in physics. If $F$ represents the force acting on a body which moves along the parametrized directed curve $x : [a, b] \rightarrow \mathbb{R}^n$, then the line integral

$$\int_C F \cdot dx$$

represent the total work performed by the force on the body. 

**Problem 6.11.** a. Suppose that $F(x, y) = xyi + (y - 3)j$ and that $C$ is the part of the parabola parametrized by

$$x : [-1, 1] \rightarrow \mathbb{R}^2, \quad x(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$$

and directed from left to right. Show that

$$F \cdot T ds = xydx + (y - 3)dy.$$

b. Find

$$\int_C F \cdot T ds.$$

It is particularly easy to calculate the line integral of a gradient along a directed curve. Indeed, the “fundamental theorem of calculus,” which asserts that differentiation and integration are inverse processes, can be generalized to the context of line integrals:
Theorem. Let \( x : [a, b] \to \mathbb{R}^2 \) be a parametrization of a directed curve \( C \) from the point \((x_0, y_0)\) to the point \((x_1, y_1)\). If \( f(x, y) \) is any smooth function, then

\[
\int_C \nabla f \cdot dx = f(x_1, y_1) - f(x_0, y_0).
\]

Problem 6.12. To prove this theorem, one uses the chain rule and the usual version of the fundamental theorem of calculus. Sketch a proof of the theorem by showing that

\[
\int_C \nabla f \cdot dx = \cdots = f(x_1, y_1) - f(x_0, y_0).
\]

The above ideas can be extended quite easily to directed curves in \( \mathbb{R}^n \). If \( x : [a, b] \to \mathbb{R}^n \) is a parametrization of a regular curve \( C \) in \( \mathbb{R}^n \) and

\[
F(x_1, \ldots, x_n) = \begin{pmatrix}
f_1(x_1, \ldots, x_n) \\
\vdots \\
f_n(x_1, \ldots, x_n)
\end{pmatrix},
\]

then the line integral

\[
\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C \mathbf{F} \cdot d\mathbf{x} = \int_C f_1 dx_1 + \cdots + f_n dx_n
\]

can be calculated by simply expressing the last integral on the right in terms of the parameter \( t \),

\[
\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_a^b \left[ f_1(x_1(t), \ldots, x_n(t)) \frac{dx_1}{dt} + \cdots + f_n(x_1(t), \ldots, x_n(t)) \frac{dx_n}{dt} \right] dt.
\]

The above theorem can also be generalized to the case where \( \mathbb{R}^2 \) is replaced by \( \mathbb{R}^n \). Thus if \( x : [a, b] \to \mathbb{R}^n \) is a parametrization of \( C \),

\[
\int_C (\nabla f) \cdot \mathbf{T} ds = f(x(b)) - f(x(a)).
\]

Problem 6.13. a. Evaluate the line integral

\[
\int_C ydx - xdy,
\]

where \( C \) is the straight line segment in \( \mathbb{R}^2 \) from \((0, 0)\) to \((3, 4)\).

b. Evaluate the same line integral in the case where \( C \) is the circle of radius one in \( \mathbb{R}^2 \) centered at the origin, directed counterclockwise.

c. Evaluate the same line integral in the case where \( C \) is the curve parametrized by

\[
x : [0, 1] \to \mathbb{R}^2, \quad \text{where} \quad x(t) = (t, t^2).
\]
Problem 6.14. a. Evaluate the line integral
\[ \int_C \nabla f \cdot dx, \]
where \( f(x, y) = x + 3y \) and \( C \) is the directed straight line segment in \( \mathbb{R}^2 \) from \((0, 0)\) to \((3, 4)\).
b. Evaluate the line integral
\[ \int_C \nabla f \cdot dx, \]
where \( f(x, y) = x + 3y \) and \( C \) is the circle of radius one in \( \mathbb{R}^2 \) centered at the origin, directed counterclockwise.
c. Evaluate the line integral
\[ \int_C \nabla f \cdot dx, \]
where \( f(x, y, z) = x^2 + y^2 + z^2 \) and \( C \) is the directed curve in \( \mathbb{R}^3 \) parametrized by
\[ x : [0, 1] \to \mathbb{R}^3, \text{ where } x(t) = (t, t^2, t^3). \]

H.6.1. Find the line integral
\[ \int_C f(x, y) ds, \]
where \( f(x, y) = x^2 + x + 1 \) and \( C \) is the circle of radius two in \( \mathbb{R}^2 \) centered at the origin.

H.6.2. a. Evaluate the line integral
\[ \int_C z dx - x dy + y dz, \]
in the case where \( C \) is the curve in \( \mathbb{R}^3 \) parametrized by
\[ x : [0, 1] \to \mathbb{R}^3, \text{ where } x(t) = (t, t^2, t^3). \]
b. Evaluate the line integral
\[ \int_C \mathbf{F} \cdot T ds, \]
where \( \mathbf{F}(x, y, z) = x \mathbf{i} + y \mathbf{j} - z \mathbf{k} \) and \( C \) is the straight line segment in \( \mathbb{R}^3 \) from \((0, 0, 0)\) to \((1, 2, 2)\).