A. Definition of double and triple integrals

Suppose that \( D \) is a bounded region in the \((x, y)\)-plane. We say that \( D \) is of type I if it can be described in the form

\[
D = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, \phi(x) \leq y \leq \psi(x)\},
\]

and of type II if it is of the form

\[
D = \{(x, y) \in \mathbb{R}^2 : c \leq y \leq d, \phi(y) \leq x \leq \psi(y)\}.
\]

We say that \( D \) is elementary if it is one of these two types.

Suppose, in addition, that \( f : D \to \mathbb{R} \) is a continuous function. If \( D \) is an elementary region of type I, we set

\[
\int \int_D f(x, y) \, dx \, dy = \int_a^b \left[ \int_{\phi(x)}^{\psi(x)} f(x, y) \, dy \right] \, dx,
\]

while if \( D \) is of type II, we set

\[
\int \int_D f(x, y) \, dx \, dy = \int_c^d \left[ \int_{\phi(y)}^{\psi(y)} f(x, y) \, dx \right] \, dy.
\]

It can be shown that if a region \( D \) is of both type I and type II, the two expressions for the double integral (1) and (2) agree. If \( D \) can be divided up into a finite union of elementary regions \( D_1, \ldots, D_k \) such that each intersection \( D_i \cap D_j \) consists of finitely many curves, then

\[
\int \int_D f(x, y) \, dx \, dy = \int \int_{D_1} f(x, y) \, dx \, dy + \cdots + \int \int_{D_k} f(x, y) \, dx \, dy.
\]

It can be shown that the result obtained is independent of the way in which \( D \) is divided up into a finite disjoint union of elementary regions.

The double integral has many possible interpretations. Some of the most important are these:
First, if \( f(x, y) \equiv 1 \), then
\[
\int \int_D 1 \, dx \, dy = \text{(area of } D)\).
\]
If \( f(x, y) \geq 0 \), then
\[
\int \int_D f(x, y) \, dx \, dy
\]
is the volume of the region
\[
E = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D, 0 \leq z \leq f(x, y)\}.
\]
If \( f(x, y) \) represents mass density at \( (x, y) \), then the double integral
\[
\int \int_D f(x, y) \, dx \, dy
\]
is the total mass of \( D \). Finally, we can use the double integral to compute the average value of \( f \) on \( D \), by means of the formula,
\[
\text{(Average value of } f \text{ on } D) = \frac{\int \int_D f(x, y) \, dx \, dy}{\int \int_D 1 \, dx \, dy}.
\]

**Problem 7.1.** Let \( D = \{(x, y) \in \mathbb{R}^2 : x^2 \leq y \leq 2 - x^2\} \).

a. Find the area of \( D \).

b. Find the center of mass of \( D \).

c. Find the average value of the function \( f(x, y) = x^2 + y^2 \) on \( D \).

d. Find the volume of the part of the paraboloid \( z = x^2 + y^2 \) which lies over \( D \).

If \( E \) is a bounded region in \((x, y, z)\)-space say that \( E \) is of type I if it can be described in the form
\[
E = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D, \phi(x, y) \leq z \leq \psi(x, y)\},
\]
for some elementary region \( D \) in the \((x, y)\)-plane. In this case
\[
\int \int \int_E f(x, y, z) \, dx \, dy \, dz = \int \int_D \left[ \int_{\phi(x, y)}^{\psi(x, y)} f(x, y, z) \, dz \right] \, dx \, dy. \tag{3}
\]
Similarly, \( E \) is of type II if it can be described in the form
\[
E = \{(x, y, z) \in \mathbb{R}^3 : (x, z) \in D, \phi(x, z) \leq y \leq \psi(x, z)\},
\]
for some elementary region \( D \) in the \((x, z)\)-plane, in which case we set
\[
\int \int \int_E f(x, y, z) \, dx \, dy \, dz = \int \int_D \left[ \int_{\phi(x, z)}^{\psi(x, z)} f(x, y, z) \, dy \right] \, dx \, dz. \tag{4}
\]
and $E$ is of type III if it can be described in the form

$$E = \{(x, y, z) \in \mathbb{R}^3 : (y, z) \in D, \phi(y, z) \leq x \leq \psi(y, z)\},$$

for some elementary region $D$ in the $(y, z)$-plane, in which case we set

$$\int \int \int_E f(x, y, z)\,dx\,dy\,dz = \int \int_D \left[ \int_{\phi(y, z)}^{\psi(y, z)} f(x, y, z)\,dx \right]\,dy\,dz. \quad (5)$$

If the three-dimensional region $E$ can be divided up into a finite union of elementary regions $E_1, \ldots, E_k$ such that each intersection $E_i \cap E_j$ consists of finitely many surfaces, then we can define the integral of $f$ over $E$ by the formula

$$\int \int \int_E f(x, y, z)\,dx\,dy\,dz = \int \int \int_{E_1} f(x, y, z)\,dx\,dy\,dz + \ldots + \int \int \int_{E_k} f(x, y, z)\,dx\,dy\,dz.$$

Like the double integral, the triple integral has many possible interpretations, depending on the context: volume, mass, average value, 

**Problem 7.2.** Let $E = \{(x, y, z) \in \mathbb{R}^3 : x + y + z \leq 1, x \geq 0, y \geq 0, z \geq 0\}$.

a. Find the volume of $E$.

b. Find the center of mass of $E$.

c. Find the average value of the function $f(x, y, z) = x^2 + y^2 + z^2$ on $E$.

**B. Differentials and Green’s Theorem**

If $\mathbf{F}(x, y) = M(x, y)i + N(x, y)j$ is a vector field on a region $D$ in the plane, where $M(x, y)$ and $N(x, y)$ are smooth functions on $D$, and $d\mathbf{x} = dx\mathbf{i} + dy\mathbf{j}$, then

$$\mathbf{F}(x, y) \cdot d\mathbf{x} = M(x, y)\,dx + N(x, y)\,dy$$

is called a differential.

In particular if $\mathbf{F} = \nabla f$, where $f(x, y)$ is a smooth scalar-valued function on $D$ then

$$\mathbf{F}(x, y) \cdot d\mathbf{x} = \frac{\partial f}{\partial x}(x, y)\,dx + \frac{\partial f}{\partial y}(x, y)\,dy.$$ 

We write

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy.$$ 

We say that a differential $M\,dx + N\,dy$ is **exact** if $M\,dx + N\,dy = df$ for some smooth function $f$. Note that if $M\,dx + N\,dy$ is exact, then

$$M = \frac{\partial f}{\partial x}, \quad N = \frac{\partial f}{\partial y} \quad \Rightarrow \quad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial M}{\partial y}.$$
Idea for solving differential equations via differentials: If we can write a differential equation in the form

\[ M \, dx + N \, dy = 0 \]

where \( M \, dx + N \, dy = df \), then the curves \( f(x, y) = c \) should be solutions to the differential equation.

**Problem 7.3.** Determine which of the following differentials are exact:

\[ x \, dy - y \, dx, \quad y \, dx + x \, dy, \quad e^y \, dx + x e^y \, dy. \]

**Problem 7.4.**

a. Write the differential equation

\[ \frac{dy}{dx} = -\frac{2xy + e^y}{x^2 + xe^y} \]

in the form \( M \, dx + N \, dy = 0 \). \( \text{(6)} \)

b. Is it true that

\[ \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} ? \]

c. find a function \( f(x, y) \) such that \( df = M \, dx + N \, dy \). Then \( f(x, y) = c \), where \( c \) is an arbitrary constant, should be the general solution to the differential equation \( (6) \).

**Remark.** This method of solving ordinary differential equations is called the method of exact differentials.

A differential \( M \, dx + N \, dy \) on a region \( D \) in the plane is said to be closed if

\[ \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \]

Note that every exact differential on a region \( D \) in the plane is closed, but we will see that there are closed differentials on some regions \( D \) which are closed but not exact!

Green’s Theorem relates double integrals to line integrals:

**Green’s Theorem.** Let \( D \) be a bounded region in the \((x, y)\)-plane, bounded by a piecewise smooth curve \( \partial D \), directed so that as it is traversed in the positive direction, the region \( D \) lies on the left. Let \( M(x, y) \, dx + N(x, y) \, dy \) be a differential on \( D \cup \partial D \) whose component functions \( M \) and \( N \) are smooth on \( D \cup \partial D \). Then

\[ \int_{\partial D} M \, dx + N \, dy = \int \int_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dxdy. \]

**Problem 7.5.** Use Green’s Theorem to evaluate the line integral

\[ \int_C (e^{-x^2} \, dx + x \, dy), \]
where \( C \) is the unit circle \( x^2 + y^2 = 1 \) traversed once in the counterclockwise direction. Hint: First reduce the line integral to a double integral and then evaluate the double integral.

Conversely, Green’s theorem is often useful in evaluating double integrals. For example, suppose we want a formula for the area of a region \( D \) bounded by a smooth closed curve \( C \). We need only find functions \( M(x, y) \) and \( N(x, y) \) so that

\[
\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1.
\]

Then

\[
\text{Area of } D = \int \int_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy = \int_C M \, dx + N \, dy.
\]

For example, we can set \( M = -y \) and \( N = 0 \), to obtain the formula

\[
\text{Area of } D = \int_C -y \, dx,
\]

or \( M = -(1/2)y \) and \( N = (1/2)x \) to obtain the formula

\[
\text{Area of } D = \int_C \left[ -(1/2)y \, dx + (1/2)x \, dy \right].
\]

(7)

**Problem 7.6.** Use Green’s Theorem to determine the area of the region in the \((x, y)\)-plane bounded by the curve \( x^{(2/3)} + y^{(2/3)} = 1 \). Hint: We can parametrize this curve by

\[
x(t) = \left( \begin{array}{c} \cos^3 t \\ \sin^3 t \end{array} \right), \quad t \in [0, 2\pi],
\]

and use formula (7). Can you sketch the curve?

**Proof of Green’s Theorem:** To prove Green’s Theorem, it suffices to prove the two simpler formulae

\[
\int_{\partial D} M \, dx = \int \int_D \left( -\frac{\partial M}{\partial y} \right) \, dx \, dy
\]

and

\[
\int_{\partial D} N \, dy = \int \int_D \left( \frac{\partial N}{\partial x} \right) \, dx \, dy.
\]

(8)

(9)

We focus on (8); the proof of (9) is similar.

To prove (8) in the case where \( D \) is of the special form

\[
D = \{ (x, y) \in \mathbb{R}^2 : a \leq x \leq b, \phi(x) \leq y \leq \psi(x) \},
\]

of type I in the terminology we used before, we note that the boundary curve \( \partial D \) divides up into four pieces:

\[
\partial D = C_1 + C_3 - C_2 - C_4,
\]
which have the following parametrizations:

- \( C_1 : x = t, y = \phi(t), a \leq x \leq b, \)
- \( C_2 : x = t, y = \psi(t), a \leq x \leq b, \)
- \( C_3 : x = a, y = t, \phi(a) \leq x \leq \psi(a), \)
- \( C_4 : x = b, y = t, \phi(b) \leq x \leq \psi(b). \)

**Problem 7.7.**

a. Show that \( dx = 0 \) along \( C_3 \) and \( C_4 \). Use this fact to evaluate

\[
\int_{C_3} M dx \quad \text{and} \quad \int_{C_4} M dx.
\]

b. Show that

\[
\int_{\partial D} M dx = \int_{C_1} M dx - \int_{C_2} M dx.
\]

c. Show that

\[
\int_{\partial D} M dx = \int_a^b M(t, \phi(t))dt - \int_a^b M(t, \psi(t))dt = \int_a^b [M(x, \phi(x)) - M(x, \psi(x))]dx.
\]

d. Use the fundamental theorem of calculus to show that

\[
\int_{\partial D} M dx = \int_a^b \int_{\phi(x)}^{\psi(x)} \left[-\frac{\partial M}{\partial y}(x, y)\right] dxdy = -\int \int_D \frac{\partial M}{\partial y}(x, y)dxdy.
\]

This establishes (8) in the case where \( D \) is of type I.

The general case of (8) is obtained by dividing a given region \( D \) into a disjoint union of regions \( D_i \) of type I. In this case,

\[
\int \int_D -\frac{\partial M}{\partial y}dxdy = \sum \int \int_{D_i} -\frac{\partial M}{\partial y}dxdy = \sum \int_{D_i} M dx = \int_{\partial D} M dx,
\]

because the parts of the boundaries of the \( D_i \)’s which lie inside \( D \) cancel in pairs.

A region \( D \subset \mathbb{R}^2 \) is said to be *convex* if

\[
p \in D \quad \text{and} \quad q \in D \quad \Rightarrow \quad (1-t)p + tq \in D \quad \text{for all} \ t \in [0,1].
\]

**Problem 7.8.** Is the region

\[
D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} - \{(0, 0)\}
\]

convex? Why or why not?

**Poincaré Lemma.** If \( D \) is a convex region in \( \mathbb{R}^2 \) then every closed differential on \( D \) is exact.
Problem 7.9. a. Suppose that $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a smooth vector field on a region $D$ in the $(x,y)$-plane, bounded by a piecewise smooth curve $\partial D$, directed so that as it is traversed in the positive direction, the region $D$ lies on the left. Let $\mathbf{T}$ denote the unit-length tangent vector to $\partial D$ and let $\mathbf{N}$ denote the outward pointing unit-length normal to $\partial D$. Show that

$$(P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}) \cdot \mathbf{N} = (-Q(x,y)\mathbf{i} + P(x,y)\mathbf{j}) \cdot \mathbf{T}$$

along $\partial D$.

b. Use Green's Theorem to prove the Divergence Theorem:

**Divergence Theorem.** Let $D$ be a bounded region in the $(x,y)$-plane, bounded by a piecewise smooth curve $\partial D$. Let $\mathbf{F}(x,y) = P(x,y)dx + Q(x,y)dy$ be a differential on $D \cup \partial D$ whose component functions $P$ and $Q$ are smooth. Then

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{N} \, ds = \int \int_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \, dx \, dy.$$ 


H.7.1. a. Suppose that

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} - \{(0,0)\}.$$ 

Show that the differential

$$Mdx + Ndy = \frac{ydx - xdy}{x^2 + y^2}$$

is closed.

b. Let $C$ be the circle $x^2 + y^2 = 1$ directed once in the counterclockwise direction. Evaluate the line integral

$$\int_C \frac{ydx - xdy}{x^2 + y^2}.$$ 

c. Does your calculation in part c show that the differential (10) is closed but not exact? Why or why not?

H.7.2. Use the Divergence Theorem to evaluate the line integral

$$\int_C \mathbf{F} \cdot \mathbf{N} \, ds,$$

where $C$ is the unit circle $x^2 + y^2 = 1$ and

$$\mathbf{F} = (y \cos e^y)\mathbf{i} + (x + y)\mathbf{j}.$$ 
