Math 6B
Introduction to Partial Differential Equations

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Preface

Partial differential equations are often used to construct models of the most basic theories underlying physics and engineering. For example, the system of partial differential equations known as Maxwell's equations can be written on the back of a post card, yet from these equations one can derive the entire theory of electricity and magnetism, which leads to an explanation of the behavior of light. But in carrying out the implications of Maxwell's equations, many of the techniques from the theory of partial differential equations prove essential.

Our goal here is to develop the most basic ideas from the theory of partial differential equations, and apply them to the simplest models arising from physics. In particular, we will present some of the elegant mathematics that could be used to describe the vibrating circular membrane, or the steady-state distribution of heat in a ball.

It is not easy to master the theory of partial differential equations. Unlike the theory of ordinary differential equations, which relies on the "fundamental existence and uniqueness theorem," there is no single theorem which is central to the subject. Instead, there are separate theories used for each of the major types of partial differential equations that commonly arise.

However, there are several basic skills which are essential for studying all types of partial differential equations. Before reading these notes, students should understand how to solve the simplest ordinary differential equations, such as the equation of exponential growth $dy/dx = ky$ and the equation of simple harmonic motion $d^2y/dx^2 + \omega y = 0$, and how these equations arise in applications, such as modeling population growth and the motion of a weight attached to the ceiling by means of a spring. Students should also understand how to solve first-order linear systems of differential equations with constant coefficients in an arbitrary number of unknowns, using vectors and matrices with real or complex entries. Familiarity is also needed with the basics of vector calculus, including the gradient, divergence and curl, and we will expand upon these ideas to present the integral theorems which explain the geometric meaning of divergence and curl. Finally, one needs ability to carry out lengthy calculations with confidence. Needless to say, all of these skills are necessary for a thorough understanding of the mathematical language that is an essential foundation for the sciences and engineering.

The subject of partial differential equations should not be studied in isolation, because much intuition comes from a thorough understanding of applications. The individual branches of the subject are concerned with the special types of partial differential equations which are needed to model diffusion, wave motion, equilibria of membranes and so forth. The behavior of physical systems often suggests theorems which can be proven via the theoretical constructs of mathematics. Conversely, the mathematics often suggests the right questions for scientists and engineers to ask.

After a first chapter which presents the integral theorems from vector calculus, we turn to two types of infinite series that are very useful for solving partial differential equations, power series in Chapter 2 and Fourier series in Chapter 3. We then treat the simplest partial differential equations in Chapter 4, the heat equation in a bar and the equation for the vibrating string. We will discuss heat and wave motion in three dimensions in Chapter 5, where the integral theorems of Chapter 1 will play a key role.

A vibrating string can be modeled quite well as a continuous object, yet if one looks at a fine enough scale, the string is made up of molecules, suggesting a discrete model with a large number of variables. The partial differential equations in the continuous model correspond to a system of ordinary differential equations in the discrete model that can be studied by the vector algebra techniques described in earlier parts of the course. Indeed, the eigenvalue problem for a differential equation thereby becomes approximated by an eigenvalue problem for an $n \times n$ matrix where n is large, providing a link between the techniques studied in linear algebra and those of partial differential equations. This correspondence can be used as the basis for numerical methods for solving the partial differential equation, and in some cases a discrete model may actually provide a better description of the phenomenon under study than a continuous one. Probabilistic techniques provide yet another component to model building, alongside the partial differential equations and discrete mechanical systems with many degrees of freedom.

There is a major dichotomy that runs through the subject of partial differential equations—linear versus nonlinear. It is actually linear partial differential equations for which the technique of linear algebra prove to be so effective. This book is concerned primarly with linear partial differential equations—yet it is the nonlinear partial differential equations that provide the most intriguing questions for research. Nonlinear partial differential equations include the Einstein field equations from general relativity and the Navier-Stokes equations which describe fluid motion, as well as many others. We hope the linear theory presented here will stimulate interest in developing an appreciation for the deeper waters of the nonlinear theory.

Doug Moore, April, 2012

One Possible Outline for a 10-week course at UCSB using these notes:

I. Integral Theorems: Chapter I, Sections 5-8, Two Weeks.

II. Power Series: Chapter 2, Sections 1-4, Two Weeks. (Discuss convergence and explain application to ODE's up to Legendre's DE. The departmental text by Lovric has additional material on convergence of series that you may want to use.)

III. Fourier Series: Chapter 3, Sections 1-3, Two Weeks.

IV. Partial Differential Equations: Chapter 4, Sections 4.1, 4.1, 4.3, 4.4, 4.5. Chapter 5, Sections 5.1, 5.2, 5.3. About Four Weeks.

If time permits one can also discuss selected topics from 5.4 through 5.7.

Contents

The sections marked with asterisks are less central to the main line of discussion, and will most likely be treated briefly or omitted if there is insufficient time.

Chapter 1

Integral Theorems from Vector Calculus

1.1 Change of variables in a multiple integral

We assume that our readers are familiar with the properties of multiple integrals, but are aware that it is sometimes difficult to evaluate such integrals. In this section, we will study a really useful trick for evaluating multiple integrals—changing the variables of integration.

Recall the useful formula for change of variable in a single integral: If

$$x = x(u), \quad u = u(x)$$

are increasing functions which give a one-to-one correspondence between $a \leq x \leq b$ and $u(a) \leq u \leq u(b)$, then

$$\int_a^b f(x)dx = \int_{u(a)}^{u(b)} f(x(u))\frac{dx}{du}du,$$

a formula which justifies the integration technique of "substitution." This formula has a nice generalization to functions of two variables. Suppose that

$$\begin{cases} x = x(u,v), \\ y = y(u,v), \end{cases} \qquad \begin{cases} u = u(x,y), \\ v = v(x,y), \end{cases}$$

give a smooth one-to-one correspondence between regions D in the (u,v)-plane and S in the (x,y)-plane. Then

$$\iint_S f(x,y)dxdy = \iint_D f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv, \tag{1.1}$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

Equation (1.1) is called the *change of variables formula* for a double integral.

To derive this formula, we use a formula for the length of a cross product which should be familiar from vector calculus:

$$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}||\mathbf{w}||\sin\theta|, \tag{1.2}$$

θ being the angle between $\mathbf{v}$ and $\mathbf{w}$. Using (1.2) it is relatively easy to show that the area of the parallelogram spanned by two vectors $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j}$ in R^2 is simply the absolute value of a determinant:

$$\text{Area} = \left|\det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}\right|.$$

Suppose now that $\mathbf{x} : D \to S$ is the mapping whose components are $x(u,v)$ and $y(u,v)$:

$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix}.$$

Let $\Box$ denote the rectangular region in D with corners at (u_0,v_0), (u_0+du,v_0), (u_0,v_0+dv), and (u_0+du,v_0+dv). The linearization of $\mathbf{x}$ at (u_0,v_0) is the affine mapping

$$\mathbf{L}(\mathbf{x}) = \mathbf{x}(u_0,v_0) + \frac{\partial \mathbf{x}}{\partial u}(u_0,v_0)(u-u_0) + \frac{\partial \mathbf{x}}{\partial v}(u_0,v_0)(v-v_0).$$

Under this affine mapping

$$(u_0,v_0) \mapsto \mathbf{x}(u_0,v_0), \quad (u_0+du,v_0) \mapsto \mathbf{x}(u_0,v_0) + \frac{\partial \mathbf{x}}{\partial u}(u_0,v_0)du,$$

$$(u_0,v_0+dv) \mapsto \mathbf{x}(u_0,v_0) + \frac{\partial \mathbf{x}}{\partial v}(u_0,v_0)dv,$$

$$(u_0+du,v_0+dv) \mapsto \mathbf{x}(u_0,v_0) + \frac{\partial \mathbf{x}}{\partial u}(u_0,v_0)du + \frac{\partial \mathbf{x}}{\partial v}(u_0,v_0)dv.$$

The four image points are the corners of the parallelogram located at $\mathbf{x}(u_0,v_0)$ and spanned by

$$\frac{\partial \mathbf{x}}{\partial u}(u_0,v_0)du \quad \text{and} \quad \frac{\partial \mathbf{x}}{\partial v}(u_0,v_0)dv.$$

According to our cross product formula, the area of this parallelogram is simply

$$\left|\frac{\partial \mathbf{x}}{\partial u}(u_0,v_0)du \times \frac{\partial \mathbf{x}}{\partial v}(u_0,v_0)dv\right| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

Since the linearization closely approximates the parametrization $\mathbf{x} : D \to R^3$ near (u_0,v_0), the area of $\mathbf{x}(\Box)$ is closely approximated by

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

Suppose first, for simplicity, that $f \equiv 1$ so that

$$\iint_S f(x,y)dxdy = \iint_S 1dxdy = \text{ Area of } S.$$

If we divide D up into many small rectangles like $\Box$ and add up their areas we obtain an approximation

$$\iint_S 1dxdy = \text{ Area of } S \doteq \sum \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv,$$

the sum being taken over all of the small rectangles. In the limit the summation becomes a double integral, and we obtain

$$\iint_S 1dxdy = \iint_D \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

This argument can be generalized to the case of a general f, in which case it yields the change of variable formula (1.1).

Example 1. Suppose that we want to find the area of the region S bounded by the ellipse $x^2 + 2xy + 5y^2 = 1$. We simply complete the squares,

$$(x+y)^2 + (2y)^2 = 1,$$

and construct a one-to-one correspondence

$$\begin{cases} u = x + y, \\ v = 2y, \end{cases} \qquad \begin{cases} x = u - \frac{1}{2}v, \\ y = \frac{1}{2}v. \end{cases}$$

It follows that

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right| = \left|\det\begin{pmatrix} 1 & -(1/2) \\ 0 & (1/2) \end{pmatrix}\right| = \frac{1}{2},$$

so

$$\text{Area of } S = \iint_{u^2+v^2\leq 1} \frac{1}{2} dudv = \frac{1}{2}(\text{area bounded by unit circle}) = \frac{\pi}{2}.$$

Here we assume as already known the fact that the area bounded by the unit circle is π, a fact we will verify in the next example.

A particularly useful special case of the change of variables formula is the case where u and v are polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta.$$

In this case,

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \left|\det\begin{pmatrix} \partial x/\partial r & \partial y/\partial r \\ \partial x/\partial \theta & \partial y/\partial \theta \end{pmatrix}\right| = \left|\det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\right| = r,$$

and hence the change of variables formula for polar coordinates is

$$\iint_S f(x,y)dxdy = \iint_D f(r\cos\theta, r\sin\theta)rdrd\theta,$$

whenever S is a region in the (x,y)-plane which corresponds in a one-to-one fashion with the region D in the (r,θ)-plane.

Example 2. Suppose that we want to use the change of variables formula to determine the area of the region S in the (x,y)-plane bounded by the circle $x^2+y^2=a^2$. In this case,

$$D = \{(r,\theta) : 0 \le r \le a,\ 0 \le \theta \le 2\pi\},$$

so

$$\text{Area of } S = \int\int_S 1dxdy = \int\int_D rdrd\theta = \left[\int_0^a r^2dr\right]\left[\int_0^{2\pi} d\theta\right] = \pi a^2.$$

Example 3. An important application of the change of variables formula is to the calculation of the area under the *Gaussian function* or *normal distribution*

$$f(u) = \frac{1}{\sqrt{2\pi}\sigma}e^{-u^2/2\sigma^2},$$

where σ is a constant known as the *standard deviation*. The graph of this probability density function is also sometimes known as the bell-shaped curve. We would like to calculate the integral

$$\int_{-\infty}^{\infty} f(u)du = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-u^2/2\sigma^2}du.$$

Now this is actually an *improper integral*, because of the infinite limits, which by definition is

$$\lim_{b\to\infty}\int_{-b}^{b}\frac{1}{\sqrt{2\pi}\sigma}e^{-u^2/2\sigma^2}du,$$

but the integral does "converge" because $f(u)$ decreases so rapidly near infinity. Being an improper integral doesn't deter us from evaluating it by a change of variables. Our first change of variables is

$$\frac{u}{\sigma} = x \quad \text{or} \quad u = \sigma x,$$

and the integral then becomes

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}dx.$$

This integral cannot be evaluated using the usual functions, such as sines, cosines and exponentials, and the cumulative distribution function

$$\Phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-x^2/2}dx$$

is tabulated in tables of mathematical functions. However, we can evaluate the entire integral

$$I = \int_{-\infty}^{\infty} e^{-x^2/2} dx$$

by first noting that

$$I^2 = \left[\int_{-\infty}^{\infty} e^{-x^2/2} dx\right] \left[\int_{-\infty}^{\infty} e^{-y^2/2} dy\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dx dy,$$

and then shifting to polar coordinates

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

We then need the fudge factor

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \partial x/\partial r & \partial y/\partial r \\ \partial x/\partial\theta & \partial y/\partial\theta \end{pmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r.$$

Applying the change of variable formula, we obtain

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dx dy = \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} r dr d\theta.$$

We finally make yet another substitution

$$v = \frac{r^2}{2}, \quad \text{so} \quad r dr = dv.$$

Then

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-v} dv d\theta = \int_0^{2\pi} \left[-e^{-v}\right]_0^{\infty} d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

We conclude that $I^2 = 2\pi$, so $I = \sqrt{2\pi}$ and

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} I = 1.$$

In other words, the integral of the Gaussian probability density function is one.

The change of variables formula can be extended to more than two variables. For example, the extension to three variables goes as follows: If

$$\begin{cases} x = x(u,v,w), \\ y = y(u,v,w), \\ z = z(u,v,w), \end{cases} \qquad \begin{cases} u = u(x,y,z), \\ v = v(x,y,z), \\ w = w(x,y,z), \end{cases}$$

give a smooth one-to-one correspondence between regions D in (u, v, w)-space and S in (x, y, z)-space,then

$$\iiint_S f(x,y,z)dxdydz$$
$$= \iiint_D f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw,$$

where

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

The proof is very similar to the two-variable case.

Example 4. Suppose, for example, that we want to calculate the volume of the region S bounded by the sphere

$$x^2 + y^2 + z^2 = a^2.$$

To do this, we use spherical coordinates

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi.$$

Then

$$\begin{pmatrix} \partial x/\partial\rho & \partial x/\partial\phi & \partial x/\partial\theta \\ \partial y/\partial\rho & \partial y/\partial\phi & \partial y/\partial\theta \\ \partial z/\partial\rho & \partial z/\partial\phi & \partial z/\partial\theta \end{pmatrix} = \begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{pmatrix},$$

and a straightforward calculation (see Exercise 1.1.3) shows that

$$\det \begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{pmatrix} = \rho^2 \sin\phi. \tag{1.3}$$

Thus if we set

$$D = \{(\rho, \phi, \theta) : 0 \le \rho \le a,\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\},$$

we find that the area of the sphere of radius a is

$$\iiint_S dxdydz = \iiint_D \rho^2 \sin\phi d\rho d\phi d\theta = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^2 \sin\phi d\rho d\phi d\theta$$
$$= \left[\int_0^a \rho^2 d\rho\right] \left[\int_0^\pi \sin\phi d\phi\right] \left[\int_0^{2\pi} d\theta\right] = \frac{a^3}{3} \cdot 2 \cdot 2\pi = \frac{4\pi a^3}{3}.$$

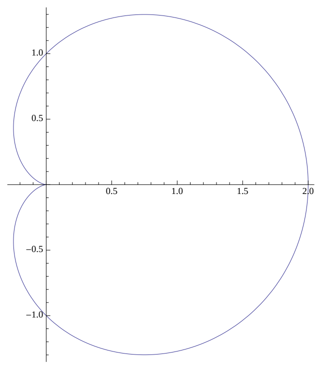

Figure 1.1: The cardioid.

Example 5. The change of variables formula for a triple integral can be used to find the volume bounded by the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$. To do this, we simply change variables by

$$\begin{cases} u = x/a, \\ v = y/b, \\ w = z/c, \end{cases}, \quad \begin{cases} x = au, \\ y = bv, \\ z = cw, \end{cases},$$

so that

$$\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

Hence using the result of Example 4, we find that

$$\begin{aligned} \text{Volume of ellipsoid} &= \iiint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leq 1} 1 dxdydz \\ &= \iiint_{x^2+y^2+z^2 \leq 1} \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dudvdw \\ &= \iiint_{x^2+y^2+z^2 \leq 1} abcdudvdw \\ &= abc(\text{Volume of unit sphere}) = \frac{4\pi abc}{3}. \end{aligned}$$

Exercises:

1.1.1. Evaluate the integral

$$\iint_D \sqrt{1 - x^2 - y^2} dxdy,$$

where $D = \{(x, y) \in R^2 : x^2 + y^2 \leq 1\}$. (Hint: Express the integrand in terms of polar coordinates.)

1.1.2. Use the change of variables formula to determine the area inside the *cardioid* $r = 1 + \cos\theta$ in the (x, y)-plane.

1.1.3. Evaluate the determinant in (1.3), thereby verifying that

$$dxdydz = \rho^2 \sin\phi d\rho d\phi d\theta.$$

1.1.4. Use spherical coordinates

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi,$$

and the change of variables formula to evaluate the triple integral

$$\int\int\int_W (x^2 + y^2)dxdydz,$$

where W is the upper half ball consisting of points satisfying the inequalities $x^2 + y^2 + z^2 \leq 4$ and $z \geq 0$. (Recall that $dxdydz = \rho^2 \sin\phi d\rho d\phi d\theta$.)

1.1.5. Use the change of variables formula to find the volume of the region in (x, y, z)-space bounded by

a. the ellipsoid $4x^2 + 9y^2 + (1/4)z^2 = 1$.
b. the ellipsoid $x^2 + 5y^2 + 10z^2 + 4xy + 2yz = 1$.
c. the surfaces $z = 4 - x^2 - 4y^2$ and $z = x^2 + 4y^2$.
d. the surfaces $x^2 + 9y^2 + z^2 = 4$ and $z = \sqrt{x^2 + 9y^2}$.

1.1.6. Use Example 3 and integration by parts to show that

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} u^2 e^{-u^2/2\sigma^2} du = \sigma^2.$$

1.2 Arc length and line integrals

Our next goal is to discuss line integrals of functions and vector fields along *regular* parametrized curves

$$\mathbf{x} : [a, b] \to R^2 \quad \text{or} \quad \mathbf{x} : [a, b] \to R^3.$$

By regular we mean that $\mathbf{x}'(t) \neq 0$ for every $t \in [a, b]$. It is easy to give examples of curves which are not regular. If

$$\mathbf{x} : [-1, 1] \to R^2 \quad \text{by} \quad \mathbf{x}(t) = \begin{pmatrix} t^2 \\ t^3 \end{pmatrix},$$

then $\mathbf{x}$ is not regular because

$$\frac{\mathbf{x}}{dt}(t) = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix} \quad \text{so} \quad \frac{\mathbf{x}}{dt}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

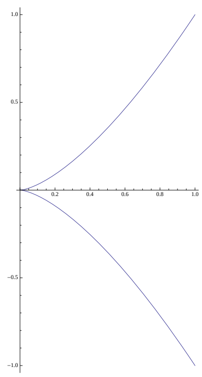

Figure 1.2: A curve which is NOT regular.

Curves which are not regular can have singular points at which the image fails to be smooth.

A key fact is that any regular smooth curve has a unit speed parametrization. By this we mean that there is a one-to-one correspondence

$$\tau : [c,d] \to [a,b], \quad t = \tau(s), \quad s = \sigma(t),$$

so that if

$$\mathbf{y}(s) = \mathbf{x}(\tau(s)), \quad \text{then} \quad \frac{d\mathbf{y}}{ds} \equiv 1.$$

In principle, we can find the arc length parameter from the formula

$$(\text{distance}) = (\text{speed})(\text{time}),$$

which can be expressed as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad \text{or} \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt,$$

but it may be hard to carry out the integration to determine the arc length s as a function of time t in practice.

An example for which we can carry out the calculation is the circle of radius five centered at the origin in R^2. The simplest parametrization of this circle is

$$\mathbf{x} : [0, 2\pi] \to R^2 \quad \text{by} \quad \mathbf{x}(t) = \begin{pmatrix} 5\cos t \\ 5\sin t \end{pmatrix}.$$

But then

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -5\sin t \\ 5\cos t \end{pmatrix},$$

$$\text{so} \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \sqrt{(-5\sin t)^2 + (5\cos t)^2} dt = 5dt,$$

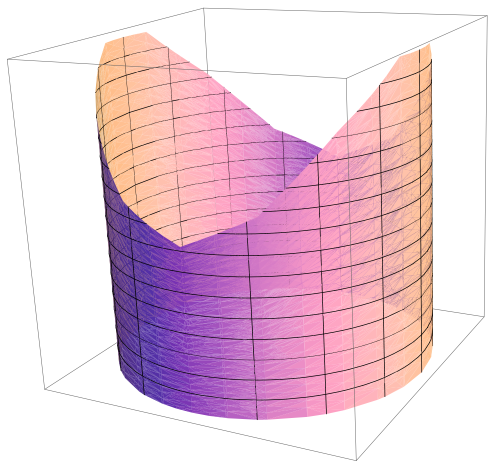

Figure 1.3: The line integral along a circle.

and we can set

$$s = 5t, \quad t = \frac{s}{5}.$$

Thus we get a new parametrization, a unit speed parametrization of the same circle,

$$\mathbf{y} : [0, 10\pi] \to R^2 \quad \text{by} \quad \mathbf{y}(s) = \begin{pmatrix} 5\cos(s/5) \\ 5\sin(s/5) \end{pmatrix}.$$

Note that the length of this circle is 10π.

Definition 1. If $\mathbf{x} : [a, b] \to R^2$ is a unit speed parametrization of a curve C in R^2 with $\mathbf{x}(s) = (x(s), y(s))$, and $f(x, y)$ is a continuously differentiable real-valued function of two variables, then

$$\int_C f ds = \int_a^b f(x(s), y(s)) ds$$

is called the line integral of f along C. Similarly, if $\mathbf{x} : [a, b] \to R^3$ is a unit speed parametrization of a curve C in R^3 with $\mathbf{x}(s) = (x(s), y(s), z(s))$, and $f(x, y, z)$ is a continuously differentiable real-valued function of three variables, then

$$\int_C f ds = \int_a^b f(x(s), y(s), z(s)) ds$$

is the line integral of f along C.

Example 1. Suppose that C is the unit circle $x^2 + y^2 = 1$ and $f(x, y) = x^2 + 2y^2$. In this case, the line integral

$$\int_C f ds \quad \text{is the area of} \quad S = \{(x, y, z) : x^2 + y^2 = 1,\ 0 \le z \le x^2 + 2y^2\}.$$

Moreover, we can use the unit speed parametrization

$$\mathbf{x} : [0, 2\pi] \to R^2 \quad \text{defined by} \quad \mathbf{x}(s) = \begin{pmatrix} \cos s \\ \sin s \end{pmatrix},$$

to evaluate this line integral. In this case,

$$x(s) = \cos s, \quad y(s) = \sin s,$$
$$f(x, y) = x^2 + 2y^2 = (\cos s)^2 + 2(\sin s)^2 = 1 + (\sin s)^2,$$

so

$$\int_C f(x(s), y(s))ds = \int_0^{2\pi} (1 + (\sin s)^2)ds = \int_0^{2\pi} \left(1 + \frac{1}{2} + \frac{1}{2}(\sin 2s)\right) ds = 3\pi.$$

A second type of line integral is often useful, the line integral of a vector field along a smooth directed or oriented curve. If

$$\mathbf{x} : [a, b] \to R^2 \quad \text{or} \quad \mathbf{x} : [a, b] \to R^3$$

is a *unit speed* parametrization of a *directed* curve $\mathbf{C}$ in R^2 or R^3, then

$$\mathbf{T}(s) = \frac{d\mathbf{x}}{ds}$$

is a unit-length vector called the *unit tangent vector* to $\mathbf{C}$.

Definition 2. The line integral of the vector field $\mathbf{F}$ along the directed curve $\mathbf{C} \subseteq R^2$ or R^3 is the line integral

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{T} ds. \tag{1.4}$$

If $\mathbf{F}$ is force, this line integral is interpreted as the *work* performed by the force on a particle that moves along the directed curve $\mathbf{C}$.

The line integral (1.4) can of course be calculated via a unit-speed parametrization for $\mathbf{C}$, but it turns out that any parametrization will do. Indeed, if

$$\mathbf{x} : [a, b] \to R^2 \quad \text{or} \quad \mathbf{x} : [a, b] \to R^3$$

is a regular parametrization in terms of the parameter t, but not of unit speed, the unit tangent vector $\mathbf{T}$ can be expressed as a function of the parameter t as

$$\mathbf{T}(t) = \frac{d\mathbf{x}/dt}{|d\mathbf{x}/dt|}.$$

Then

$$\mathbf{T} ds = \mathbf{T} \frac{ds}{dt} dt = \frac{d\mathbf{x}/dt}{|d\mathbf{x}/dt|} \frac{ds}{dt} dt = \frac{d\mathbf{x}/dt}{ds/dt} \frac{ds}{dt} dt = \frac{d\mathbf{x}}{dt} dt = d\mathbf{x},$$

so we can write

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{T} ds = \int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{x},$$

and the integral on the right can be evaluated in terms of the given parametrization.

Example 2. Suppose that $\mathbf{C}$ is the directed curve parametrized by

$$\mathbf{x} : [0,1] \to R^3 \quad \text{where} \quad \mathbf{x}(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$$

and

$$\mathbf{F}(x,y,z) = y\mathbf{i} - 2x\mathbf{j} + 6\mathbf{k}.$$

To evaluate the line integral, we write the integrand as a "differential,"

$$\mathbf{F} \cdot d\mathbf{x} = ydx - 2xdy + 6dz,$$

then replace everything in the integrand by its expression in terms of the parameter t, and integrate with respect to the limits on the parameter:

$$\begin{aligned} \int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{T} ds = \int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{x} & \\ = \int_0^1 (t^2)dt - 2td(t^2) + 6d(t^3) & \\ = \int_0^1 (t^2 - 4t^2 + 18t^2)dt = \left[5t^3\right]_0^1 = 5. & \end{aligned}$$

One can show via the chain rule that any other parametrization would give the same value for the line integral.

Exercises:

1.2.1.a. Suppose that $\mathbf{x} : [0, 2\pi] \to R^3$ is the parametrization of the part C of the helix defined by

$$\mathbf{x}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}.$$

Find a reparametrization of C which has unit speed.

b. Find the length of C.

c. Determine the line integral

$$\int_C z ds.$$

1.2.2.a. Find a parametrization of the directed line segment $\mathbf{C}$ in R^3 from the point $(1, 0, 0)$ to the point $(0, 3, 4)$.

b. Use the parametrization to calculate the line integral

$$\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{x}, \quad \text{where} \quad \mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}.$$

1.3 Conservative forces

According to Newton's law of gravitation, the gravitational force $\mathbf{F}$ which the sun located at the origin $(0,0,0)$ exerts on a planet located at the point $\mathbf{x} = (x,y,z)$ is given by the formula

$$\mathbf{F} = \frac{-GMm}{x^2+y^2+z^2} \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \frac{-GMm}{\|\mathbf{x}\|^2} \frac{\mathbf{x}}{\|\mathbf{x}\|},$$

In this formula, M is the mass of the sun, m is the mass of the planet and G is a constant (called Newton's gravitation constant). Combining this with Newton's second law of motion yields the second-order vector differential equation which governs planetary motion:

$$m\frac{d^2\mathbf{x}}{dt^2} = \frac{-GMm}{\|\mathbf{x}\|^2} \frac{\mathbf{x}}{\|\mathbf{x}\|}. \tag{1.5}$$

One of Newton's major successes was to actually solve this differential equation, which he then used to verify Kepler's three laws of planetary motion. One of those laws says that the planets trace out ellipses with the sun at one of the foci.

There are several steps to finding the solutions to (1.5). First one shows that the motion lies in a plane. Second, one shows that energy and angular momentum are constant. Finally, one shifts to polar coordinates in the plane. From here it is not too hard to apply techniques for solving first-order ODE's to derive the explicit solution. We will not carry out the details of this verification here, but point out that conservation of energy leads to an important concept, the notion of conservative force.

To show that energy is conserved in planetary motion, one first verifies that gravitational force can be derived from a potential energy function

$$V(x,y,z) = \frac{-GMm}{\sqrt{x^2+y^2+z^2}}.$$

Indeed, we can differentiate

$$V(x,y,z) = -GMm(x^2+y^2+z^2)^{-1/2},$$

obtaining

$$\frac{\partial V}{\partial x} = (-GMm)\frac{-1}{2}(x^2+y^2+z^2)^{-3/2}(2x) = \frac{GMmx}{(x^2+y^2+z^2)^{3/2}},$$

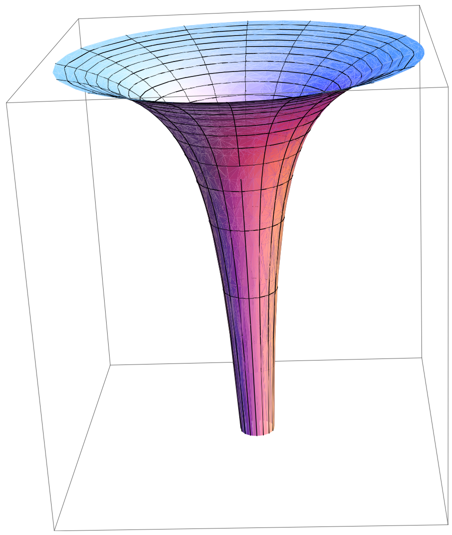

Figure 1.4: Graph of the gravitational potential with z suppressed.

$$\frac{\partial V}{\partial y} = \frac{GMmy}{(x^2+y^2+z^2)^{3/2}}, \qquad \frac{\partial V}{\partial z} = \frac{GMmz}{(x^2+y^2+z^2)^{3/2}}.$$

We can use the gradient operator ∇ to write this as

$$-\nabla V = \frac{-GMm}{x^2+y^2+z^2} \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \mathbf{F}.$$

In applying this to Newton's differential equation (1.5), we would then go on to show that for any solution, the kinetic energy plus potential energy is constant, or

$$\frac{1}{2}m\left|\frac{d\mathbf{x}}{dt}\right|^2 + V(x,y,z) = \text{constant}.$$

The Newtonian force $\mathbf{F}$ is an important example of a vector field which is conservative:

Definition. A smooth vector field $\mathbf{F}$ on R^3 is *conservative* if

$$\mathbf{F} = -\nabla V = -\frac{\partial V}{\partial x}\mathbf{i} - \frac{\partial V}{\partial y}\mathbf{j} - \frac{\partial V}{\partial z}\mathbf{k}$$

for some smooth function V on R^3, or equivalently, if $\mathbf{F} = \nabla\phi$, for some smooth function ϕ.

It turns out that line integrals of conservative vector fields are easily calculated:

Fundamental Theorem of Calculus. Suppose that $\mathbf{x} : [a,b] \to R^2$ is a parametrization of a regular directed curve $\mathbf{C}$ from (x_a, y_a, z_a) to (x_b, y_b, z_b)

and that $\mathbf{F}(x,y,z) = \nabla\phi(x,y,z)$, where $\phi(x,y,z)$ is a smooth function. Then

$$\int_{\mathbf{C}} \mathbf{F}\cdot\mathbf{T}ds = \int_{\mathbf{C}} \mathbf{F}\cdot d\mathbf{x} = \phi(x_b,y_b,z_b) - \phi(x_a,y_a,z_a).$$

The theorem implies that the line integral depends **only on the endpoints**, not the shape of $\mathbf{C}$. The theorem holds because of the chain rule:

$$\begin{aligned}\int_{\mathbf{C}} \mathbf{F}\cdot\mathbf{T}ds &= \int_{\mathbf{C}} \mathbf{F}\cdot d\mathbf{x} = \int_{\mathbf{C}} \nabla\phi\cdot d\mathbf{x}\\ &= \int_{\mathbf{C}} \nabla\phi\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k}) = \int_{\mathbf{C}} \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\\ &= \int_a^b \frac{\partial\phi}{\partial x}\frac{dx}{dt}dt + \frac{\partial\phi}{\partial y}\frac{dy}{dt}dt + \frac{\partial\phi}{\partial z}\frac{dz}{dt}dt\\ &= \int_a^b \frac{d}{dt}(\phi(x(t),y(t),z(t))dt = \phi(x_b,y_b,z_b) - \phi(x_a,y_a,z_a).\end{aligned}$$

For example, suppose that a planet is moving around the sun with a Newtonian gravitational potential

$$V(x,y,z) = \frac{-GMm}{\sqrt{x^2+y^2+z^2}} \quad \text{so that} \quad \mathbf{F} = -\nabla V = \nabla\phi,$$

where $\phi = -V$. Recalling that work is just the line integral of force, we see that the work performed on the planet by the gravitational field when the planet moves along a directed curve $\mathbf{C}$ is

$$(\text{Work}) = \int_{\mathbf{C}} \nabla\phi\cdot\mathbf{T}ds = -\int_{\mathbf{C}} \nabla V\cdot\mathbf{T}ds = -\int_{\mathbf{C}} \nabla V\cdot d\mathbf{x}.$$

If $\mathbf{C}$ is any directed curve from $(3,4,0)$ to $(0,0,1)$, it follows from the Fundamental Theorem of Calculus that the work that the gravitational field performs on the planet as it moves along this curve is

$$\text{Work} = V(3,4,0) - V(0,0,1) = \frac{-GMm}{5} + GMm = \frac{4}{5}GMm.$$

More generally, if $\mathbf{C}$ is any directed curve from (x_a,y_a,z_a) to (x_b,y_b,z_b), the work performed by the gravitational field on a planet moving along $\mathbf{C}$ is just

$$V(x_a,y_a,z_a) - V(x_b,y_b,z_b),$$

the difference in potential energies.

Exercise:

1.3.1. Evaluate the line integral

$$\int_{\mathbf{C}} \nabla f\cdot d\mathbf{x},$$

where

a. $f(x,y,z) = x + 3y + 2z$ and $\mathbf{C}$ is the directed straight line segment in R^3 from $(0,0,0)$ to $(1,3,4)$.

b. $f(x,y,z) = x^2 + y^2 + z^2$ and $\mathbf{C}$ is the directed curve in R^3 parametrized by

$$\mathbf{x} : [0,1] \to R^3, \quad \text{where} \quad \mathbf{x}(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}.$$

1.4 Green's Theorem

Given a vector field $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ on the plane, we write

$$d\mathbf{x} = dx\mathbf{i} + dy\mathbf{j}.$$

and

$$\mathbf{F} \cdot \mathbf{T} ds = \mathbf{F} \cdot d\mathbf{x} = M(x,y)dx + N(x,y)dy.$$

This last expression is often called a **differential**, which is just another name for an integrand for a line integral.

Green's Theorem. Suppose that D is a bounded region in the (x,y)-plane, a region bounded by a piecewise smooth directed curve $\mathbf{C}$, the direction chosen so that as $\mathbf{C}$ is traversed in the positive direction, D is on the left. If $M(x,y)$ and $N(x,y)$ are well behaved on D, then

$$\int_{\mathbf{C}} M dx + N dy = \int \int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy.$$

Alternate form: If the vector field

$$\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$$

is well-behaved on D, then

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{T} ds = \int \int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy.$$

For example, suppose that $\mathbf{C}$ is the counterclockwise ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1,$$

where a and b are positive constants, traversed in the counterclockwise direction. We would like to calculate the line integral

$$\int_{\mathbf{C}} (-y dx + x dy).$$

Method I. To calculate the integral directly, we need a parametrization of the ellipse. One such parametrization is

$$x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi.$$

We can then write

$$\begin{aligned} dx = -a\sin t dt, \quad & dy = b\cos t dt, \\ & (-ydx + xdy) = -(b\sin t)(-a\sin t dt) + (a\cos t)(b\cos t dt) = ab dt. \end{aligned}$$

Thus

$$\int_{\mathbf{C}} (-ydx + xdy) = \int_0^{2\pi} ab dt = 2\pi ab.$$

Method II. Instead, we can apply Green's Thoerem: If

$$D = \left\{ (x, y) : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \le 1 \right\},$$

then

$$\begin{aligned} \int_{\mathbf{C}} (-ydx + xdy) &= \int\int_D \left(\frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y} \right) dxdy \\ &= \int\int_D 2dxdy = 2(\text{area of } D) = 2\pi ab. \end{aligned}$$

The last step is obtained by the formula for change of variables in a double integral.

Here is another example: If $\mathbf{C}$ is the counterclockwise unit circle, what is the line integral

$$\int_{\mathbf{C}} (e^{-x^2} dx + xdy)?$$

In this case, direct integration of the line integral looks difficult, but it is easy to do the calculation via Green's Theorem where $M = e^{-x^2}$ and $N = x$:

$$\begin{aligned} \int_{\mathbf{C}} (e^{-x^2} dx + xdy) &= \int_{\mathbf{C}} Mdx + Ndy \\ &= \int\int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy \\ \int\int_D & \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(e^{-x^2}) \right] dxdy \\ &= \int\int_D [1 - 0] dxdy = \int\int_D dxdy = \pi. \end{aligned}$$

We can also use Green's Theorem to calculate the area of a region D bounded by a piecewise smooth curve $\mathbf{C}$, which is transversed counterclockwise. Just set

$$M = -\frac{1}{2}y, \qquad N = \frac{1}{2}x.$$

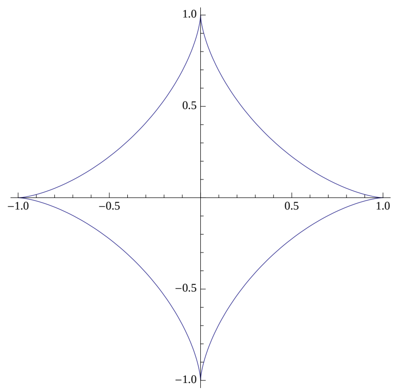

Figure 1.5: The curve $x^{(2/3)} + y^{(2/3)} = 1$.

Then

$$\begin{aligned}\int_{\mathbf{C}} M dx + N dy &= \int\int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy \\ &= \int\int_D \left(\frac{\partial}{\partial x} \left(\frac{1}{2} x \right) - \frac{\partial}{\partial y} \left(-\frac{1}{2} y \right) \right) dxdy \\ &= \int\int_D dxdy = \text{Area of } D.\end{aligned}$$

For example, suppose that we want to find the area of the region D bounded by the curve

$$x^{(2/3)} + y^{(2/3)} = 1.$$

We can parametrize this by

$$x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le 2\pi,$$

$$dx = -3\cos^2 t \sin t dt, \quad dy = 3\sin^2 t \cos t dt,$$

$$\begin{aligned}-ydx + xdy &= 3\sin^4 t \cos^2 t dt + 3\sin^2 t \cos^4 t dt \\ &= 3\sin^2 t \cos^2 t dt = \frac{3}{4}(\sin 2t)^2 dt = \frac{3}{8}(1 - \cos 4t) dt.\end{aligned}$$

The result is

$$\text{Area of } D = \int_{\mathbf{C}} -\frac{1}{2} ydx + \frac{1}{2} xdy = \int_0^{2\pi} \frac{3}{16}(1 - \cos 4t) dt = \cdots = \frac{3}{8}\pi.$$

Closely related to Green's Theorem is:

Divergence Theorem in the Plane. Suppose that D is a bounded region in the (x, y)-plane, a region bounded by a piecewise smooth directed curve $\mathbf{C}$. Let $\mathbf{N}$ be the outward-pointing unit-normal along $\mathbf{C}$. If

$$\mathbf{V}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

is a well-behaved vector field on D, then

$$\int_{\mathbf{C}} \mathbf{V} \cdot \mathbf{N} ds = \int\int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx dy.$$

Here the function

$$\nabla \cdot \mathbf{V} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

is called the *divergence* of $\mathbf{V}$.

Actually, this is just another version of Green's Theorem. Indeed, we could let $\star$ denote counterclockwise rotation through 90 degrees. Then $\star\mathbf{N} = \mathbf{T}$ and

$$\star(P\mathbf{i} + Q\mathbf{j}) = -Q\mathbf{i} + P\mathbf{j},$$

so we can apply Green's Theorem to obtain

$$\begin{aligned}\int_{\mathbf{C}} \mathbf{V} \cdot \mathbf{N} ds &= \int_{\mathbf{C}} (\star\mathbf{V}) \cdot (\star\mathbf{N}) ds = \int_{\mathbf{C}} (-Q\mathbf{i} + P\mathbf{j}) \cdot \mathbf{T} ds \\ &= \int_{\mathbf{C}} -Q dx + P dy = \int\int_D \left(\frac{\partial P}{\partial x} - \frac{\partial (-Q)}{\partial y} \right) dx dy \\ &= \int\int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx dy.\end{aligned}$$

For example, suppose that

$$\mathbf{F}(x, y) = (y \cos e^y)\mathbf{i} + (x + y^2)\mathbf{j}$$

and that $\mathbf{C}$ is the counterclockwise boundary of the triangular region

$$D = \{(x, y) : 0 \le x,\ 0 \le y,\ x + y \le 1\}.$$

We can ask: What is

$$\int_{\mathbf{C}} \mathbf{V} \cdot \mathbf{N} ds?$$

It would be quite difficult to evaluate the line integral directly, but the calculation via the divergence theorem is easy.

First, we note that

$$\nabla \cdot \mathbf{F}(x, y) = \frac{\partial}{\partial x}(y \ \cos e^y) + \frac{\partial}{\partial y}(x + y^2) = 0 + 2y = 2y.$$

Thus

$$\int_{\mathbf{C}} \mathbf{V} \cdot \mathbf{N} ds = \iint_D \nabla \cdot \mathbf{F}(x, y) dx dy = \iint_D 2y dx dy.$$

We can now calculate the integral, obtaining

$$\int_{\mathbf{C}} \mathbf{V} \cdot \mathbf{N} ds = \int_0^1 \left[\int_0^{1-x} 2y dy \right] dx = \cdots = \frac{1}{3}.$$

Exercises:

1.4.1. Use Green's theorem (and the formula for changing variables in a multlple integral if needed) to evaluate the line integral

$$\int_C (e^{-x^2} - 2x^2 y)dx + 2xy^2 dy,$$

where C is the unit circle traversed counterclockwise.

1.4.2. Suppose that $\mathbf{C}$ is the boundary of the rectangle

$$0 \leq x \leq 1, \quad 0 \leq y \leq 2,$$

and that $\mathbf{F} = (x - e^y, \sin(x^2 + 4))$. If $\mathbf{N}$ is the outward-pointing unit normal, use the Divergence Theorem in the plane to calculate

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{N} ds.$$

1.5 Surface area and surface integrals

Sometimes it is convenient to represent a surface S in R^3 as the image of smooth vector-valued function, which is called a parametrization of S.

For example, the *paraboloid of revolution* in R^3, defined as the set of points which satisfy the equation

$$z = x^2 + y^2,$$

can be thought of as the graph of the function $f(x, y) = x^2 + y^2$. But we can also regard it as the image of the vector-valued function

$$\mathbf{x} : R^2 \to R^3 \quad \text{which is defined by} \quad \mathbf{x}(u, v) = \begin{pmatrix} u \\ v \\ u^2 + v^2 \end{pmatrix}. \tag{1.6}$$

This function $\mathbf{x}$ is called a parametrization for the paraboloid of revolution.

In general, a *parametrization* of a smooth surface S is simply a smooth one-to-one vector-valued function $\mathbf{x}$ from a domain D in the (u, v) plane onto S. We can use parametrizations to find areas of surfaces, and more generally calculate surface integrals.

We first consider the problem of calculating surface area, and start with the observation that the area of the parallelogram spanned by two vectors $\mathbf{v}$ and $\mathbf{w}$ is simply the length of their cross product,

$$\text{Area} = |\mathbf{v} \times \mathbf{w}|.$$

Suppose now that $\mathbf{x} : D \to R^3$ is the parametrization of a surface S, and that $(u_0, v_0) \in D$. Let $\Box$ denote the rectangular region in D with corners at

(u_0, v_0), $(u_0 + du, v_0)$, $(u_0, v_0 + dv)$, and $(u_0 + du, v_0 + dv)$. The linearization of $\mathbf{x}$ at (u_0, v_0) is the affine mapping

$$\mathbf{L}(\mathbf{x}) = \mathbf{x}(u_0, v_0) + \frac{\partial \mathbf{x}}{\partial u}(u - u_0) + \frac{\partial \mathbf{x}}{\partial v}(v - v_0).$$

Under this affine mapping

$$(u_0, v_0) \mapsto \mathbf{x}(u_0, v_0), \quad (u_0 + du, v_0) \mapsto \mathbf{x}(u_0, v_0) + \frac{\partial \mathbf{x}}{\partial u} du,$$

$$(u_0, v_0 + dv) \mapsto \mathbf{x}(u_0, v_0) + \frac{\partial \mathbf{x}}{\partial v} dv, \quad (u_0 + du, v_0 + dv) \mapsto \mathbf{x}(u_0, v_0) + \frac{\partial \mathbf{x}}{\partial u} du + \frac{\partial \mathbf{x}}{\partial v} dv.$$

The four image points are the corners of the parallelogram located at $\mathbf{x}(u_0, v_0)$ and spanned by

$$\frac{\partial \mathbf{x}}{\partial u}(u_0, v_0) du \quad \text{and} \quad \frac{\partial \mathbf{x}}{\partial v}(u_0, v_0) dv,$$

a parallelogram which has area

$$dA = \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| dudv.$$

Since the linearization closely approximates the parametrization $\mathbf{x} : D \to R^3$ near (u_0, v_0), the area of $\mathbf{x}(\Box)$ is closely approximated by

$$dA = \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| dudv.$$

If we divide D up into many small rectangles like $\Box$ and add up their contributions to the area, we obtain the following formula for the surface area of a surface S parametrized by $\mathbf{x} : D \to R^3$:

$$\text{Surface area of } S = \iint_D \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| dudv.$$

Example 1. Suppose that we want to find the area of the part of the paraboloid of revolution (1.6) which lies within the cylinder $x^2 + y^2 = 1$. In this case, we can take the parametrization

$$\mathbf{x}(u, v) = \begin{pmatrix} u \\ v \\ u^2 + v^2 \end{pmatrix}, \quad u^2 + v^2 \le 1,$$

and we find that

$$\frac{\partial \mathbf{x}}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix}, \quad \frac{\partial \mathbf{x}}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ 2v \end{pmatrix},$$

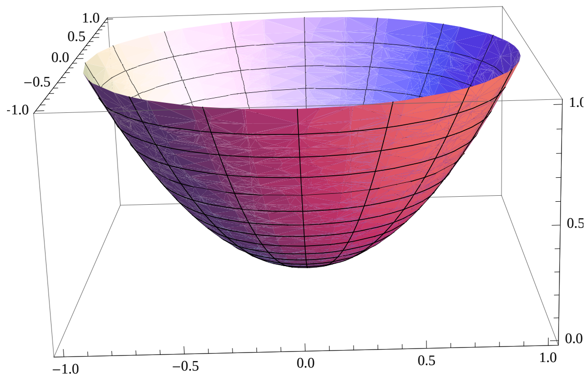

Figure 1.6: The paraboloid of revolution.

$$\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = \begin{pmatrix} -2u \\ -2v \\ 1 \end{pmatrix}.$$

$$\left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| = \sqrt{1 + 4u^2 + 4v^2}, \quad dA = \sqrt{1 + 4u^2 + 4v^2}\ dudv.$$

so that

$$\text{Area of } S = \iint_D \sqrt{1 + 4u^2 + 4v^2}\ dudv,$$

$$\text{where} \quad D = \{(u, v) : u^2 + v^2 \leq 1\}.$$

To carry out the integration, we transform to polar coordinates, and obtain

$$\text{Area of } S = \iint_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2}\ rdrd\theta = 2\pi \int_0^1 \frac{1}{8}(1 + 4r^2)^{1/2}(8rdr),$$

which simplifies to yield

$$\text{Area of } S = \frac{\pi}{4}\frac{2}{3}(1 + 4r^2)^{3/2}\Big|_0^1 = \frac{\pi}{6}(5\sqrt{5} - 1).$$

More generally, if $\mathbf{x} : D \to R^3$ is the parametrization of a surface $\mathbf{S}$ and $f(x, y, z)$ is any continuous real-valued function of three variables, the *surface integral* of f over $\mathbf{S}$ is given by the formula

$$\iint_{\mathbf{S}} f(x, y, z) dA = \int \int_D f(x(u, v), y(u, v), z(u, v)) \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| dudv.$$

In more advanced texts it is shown that the integral thus defined is independent of parametrization.

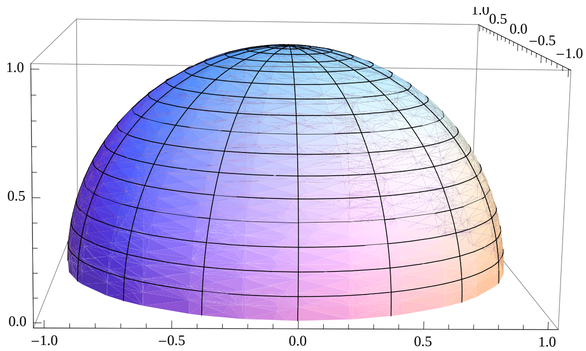

Figure 1.7: The upper hemisphere.

This surface integral can be used for many purposes. Thus if $f(x, y, z)$ represents mass per unit are on the surface $\mathbf{S}$, then

$$\iint_{\mathbf{S}} f(x, y, z)dA$$

represents the total mass of the surface. More generally, if $f(x, y, z)$ is any smooth function,

$$\text{the average value of } f \text{ on } S = \frac{\int_S f dA}{\int_S 1dA}.$$

For example, suppose that we want to find the center of mass of a surface S, assuming that it is made of a material which has constant density per unit area. Then the coordinates of the center of mass are $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{\iint_S xdA}{\iint_S 1dA}, \quad \bar{y} = \frac{\iint_S ydA}{\iint_S 1dA}, \quad \bar{z} = \frac{\iint_S zdA}{\iint_S 1dA}.$$

Example 2. Suppose that we want to find the center of mass of the hemisphere S defined by

$$x^2 + y^2 + z^2 = 1, \qquad z \geq 0.$$

Symmetry implies that $\bar{x} = 0 = \bar{y}$, so we need only determine $\bar{z}$. To calculate $\bar{z}$, it is expedient to use spherical coordinates:

$$\begin{cases} x = \rho \sin\phi \cos\theta, \\ y = \rho \sin\phi \sin\theta, \\ z = \rho \cos\phi, \end{cases} \qquad \begin{cases} x = \sin u \cos v, \\ y = \sin u \sin v, \\ z = \cos u, \end{cases}$$

where $0 \leq u \leq \pi/2$, $0 \leq v \leq 2\pi$. This provides the parametrization $\mathbf{x} : D \to R^3$, where

$$D = \{(u, v) : 0 \leq u \leq \pi/2, \ 0 \leq v \leq 2\pi\} \quad \text{and} \quad \mathbf{x}(u, v) = \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix}.$$

Thus

$$\frac{\partial \mathbf{x}}{\partial u} = \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ -\sin u \end{pmatrix}, \quad \frac{\partial \mathbf{x}}{\partial v} = \begin{pmatrix} -\sin u \sin v \\ \sin u \cos v \\ 0 \end{pmatrix},$$

$$\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} = \sin u \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos u \cos v & \cos u \sin v & -\sin u \\ -\sin v & \cos v & 0 \end{vmatrix} = \sin u \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix},$$

$$\left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| = |\sin u|, \quad dA = \sin u \; dudv,$$

and we conclude that

$$\text{Area of } S = \iint_S dA = \int_0^{2\pi} \int_0^{\pi/2} \sin u \; dudv = \cdots = 2\pi.$$

Since $z = \cos u$,

$$\iint_S z dA = \int_0^{2\pi} \int_0^{\pi/2} \cos u \sin u \; dudv$$
$$= \int_0^{2\pi} \left[\frac{1}{2} \sin^2 u \right]_0^{\pi/2} dv = 2\pi \cdot \frac{1}{2} = \pi,$$

and hence

$$\bar{z} = \frac{\iint_S z dA}{\iint_S 1 dA} = \frac{\pi}{2\pi} = \frac{1}{2},$$

so the center of mass is $(0, 0, 1/2)$.

Exercises:

1.5.1. Calculate the surface integral

$$\iint_S z^2 dA,$$

where S is the paraboloid considered in Example 1.

1.5.2.a. Calculate the area of the sphere

$$x^2 + y^2 + z^2 = a^2$$

by using the parametrization $\mathbf{x} : D \to R^3$, where

$$D = \{(u, v) : 0 \leq u \leq \pi, \; 0 \leq v \leq 2\pi\}, \qquad \mathbf{x}(u, v) = \begin{pmatrix} a \sin u \cos v \\ a \sin u \sin v \\ a \cos u \end{pmatrix}.$$

b. Calculate the surface integral

$$\iint_S z^2 dA,$$

where S is the surface considered in part a.

1.5.3. Find a parametrization for the part of the cylinder $x^2 + y^2 = 1$ which lies between the planes $z = -1$ and $z = 1$. (Hint: Use polar coordinates in the (s, y)-plane.) Use the parametrization to find the area of this part of the cylinder.

1.5.4. Let S be the *torus* defined by the equation

$$(\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1,$$

with the parametrization $\mathbf{x} : D \to S$, defined by

$$\mathbf{x}(u, v) = \begin{pmatrix} (2 + \cos v) \cos u \\ (2 + \cos v) \sin u \\ \sin v \end{pmatrix},$$

where

$$D = \{(u, v) \in R^2 : -\pi < u < \pi, -\pi < v < \pi\}.$$

Find the surface area of S.

1.6 Flux integrals

The flux integral provides an important application of the surface integral described in the previous section. Suppose that we have a continuous choice of unit-normal $\mathbf{N}$ to a smooth surface $\mathbf{S}$. Such a continuous choice of unit-normal is called an *orientation* of $\mathbf{S}$. If

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

is a smooth vector field on R^3, then the *flux* of $\mathbf{F}$ through $\mathbf{S}$ is given by the surface integral

$$\iint_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N} dA,$$

the integrand being calculated by the same method as in the previous section

At first, one might think that calculation of flux integrals is difficult. But calculation of flux integrals is simpler than might be expected, because

$$\mathbf{N} dA = \frac{\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}}{\left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right|} \left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right| dudv = \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} dudv,$$

and hence

$$\iint_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N} dA = \iint_D \mathbf{F}(x(u, v), y(u, v), z(u, v)) \cdot \left(\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right) dudv.$$

Given a parametrization $\mathbf{X} : D \to S$, the integral on the right is often rather routine.

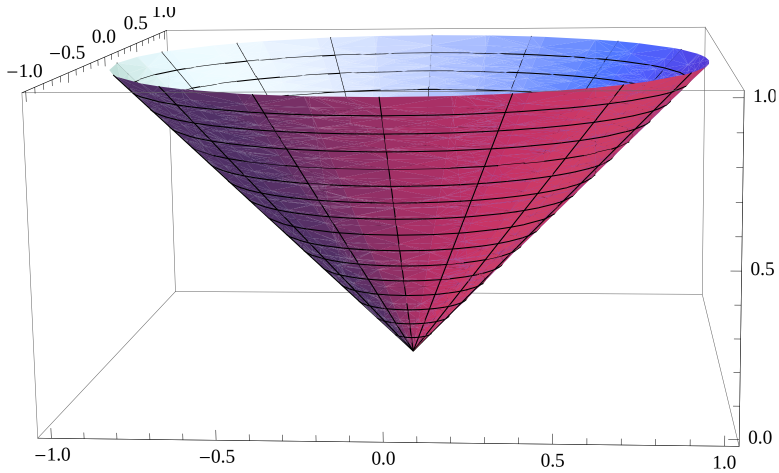

Figure 1.8: The half cone.

Example. For example, suppose that S is the part of the half-cone defined by

$$z^2 = x^2 + y^2, \qquad z \geq 0, \qquad x^2 + y^2 \leq 1.$$

Using polar coordinates $x = u\cos v$, $y = u \sin v$ in the (x, y)-plane, we can parametrize this surface by

$$D = \{(u, v) : 0 \leq u \leq 1,\ 0 \leq v \leq 2\pi\}, \qquad \mathbf{x}(u, v) = \begin{pmatrix} u\cos v \\ u \sin v \\ u \end{pmatrix}.$$

In this case,

$$\frac{\partial \mathbf{x}}{\partial u} = \begin{pmatrix} \cos v \\ \sin v \\ 1 \end{pmatrix}, \quad \frac{\partial \mathbf{x}}{\partial v} = \begin{pmatrix} -u\sin v \\ u\cos v \\ 0 \end{pmatrix},$$

$$\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 1 \\ -u \sin v & u\cos v & 0 \end{vmatrix} = \begin{pmatrix} -u\cos v \\ -u\sin v \\ u \end{pmatrix}.$$

$$\left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| = \sqrt{2}u, \quad dA = \sqrt{2}u\ dudv.$$

Thus the area of S is given by

$$\iint_S dA = \int_0^{2\pi} \int_0^1 \sqrt{2}u\ dudv = \int_0^{2\pi} \left[\frac{\sqrt{2}}{2} u^2 \right]_0^1 dv = \sqrt{2}\pi.$$

To calculate the flux integral

$$\iint_S \mathbf{F} \cdot \mathbf{N} dA,$$

where

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}.$$

We can write the vector field in terms of u and v as

$$(\mathbf{F} \circ \mathbf{x})(u, v) = \begin{pmatrix} u\cos v \\ u\sin v \\ 2u \end{pmatrix}.$$

Since

$$\mathbf{N}dA = \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} dudv = \begin{pmatrix} -u\cos v \\ -u\sin v \\ u \end{pmatrix} dudv,$$

we see that

$$\mathbf{F} \cdot \mathbf{N}dA = (-u^2\cos^2 v - u^2\sin^2 v + 2u^2)dudv = u^2 dudv.$$

So

$$\iint_S \mathbf{F} \cdot \mathbf{N}dA = \int_0^{2\pi} \int_0^1 u^2 \; dudv = \int_0^{2\pi} \left[\frac{1}{3}u^3\right]_0^1 dv = \frac{2\pi}{3}.$$

A physical picture for the flux integral: Suppose that a fluid is flowing throughout (x, y, z)-space with velocity $\mathbf{V}(x, y, z)$ and density $\rho(x, y, z)$. In this case, fluid flow is represented by the vector field

$$\mathbf{F} = \rho\mathbf{V},$$

and the surface integral

$$\iint_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N}dA \tag{1.7}$$

represents the rate at which the fluid is flowing across $\mathbf{S}$ in the direction of $\mathbf{N}$. Indeed, the rate at which fluid flows across a small piece of $\mathbf{S}$ of surface area dA is

$$(\text{density})(\text{normal component of velocity})dA = \rho\mathbf{V} \cdot \mathbf{N}dA.$$

If we add up the contributions of all the small area elements, we obtain the integral (1.7).

Exercises:

1.6.1.a. Suppose that S is the paraboloid

$$z = 4 - (x^2 + y^2), \qquad x^2 + y^2 \leq 4.$$

Find a region D in the (u, v)-plane and a vector-valued function $\mathbf{x} : D \to R^3$ which parametrizes S.

b. Using the formula

$$dA = \left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right| dudv,$$

determine the surface area of S.

c. If $\mathbf{F} = z\mathbf{k}$, use the formula

$$\mathbf{N}dA = \left(\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right) dudv$$

to evaluate the flux integral

$$\iint_S \mathbf{F} \cdot \mathbf{N}dA.$$

1.6.2.a. Show that $\mathbf{x} : D \to R^3$, where

$$D = \{(z,\theta) : -1 \leq z \leq 1,\ 0 \leq \theta \leq 2\pi\} \quad \text{and} \quad \mathbf{x}(z,\theta) = \begin{pmatrix} \sqrt{z^2+1}\cos\theta \\ \sqrt{z^2+1}\sin\theta \\ z \end{pmatrix},$$

is a parametrization of the part of the hyperboloid of one sheet $x^2+y^2-z^2=1$ which lies between the planes $z=-1$ and $z=1$.

b. Evaluate the flux integral

$$\iint_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N}dA,$$

where $\mathbf{S}$ is the part of the hyperboloid of one sheet $x^2+y^2-z^2=1$ which lies between $z=-1$ and $z=1$, $\mathbf{F}(x,y,z) = (x,y,0)$, and $\mathbf{N}$ is the outward-pointing unit normal.

1.7 The Divergence Theorem

We now turn to the integral theorems in three dimensions, and this involves the three key operations—gradient, divergence and curl—which are all defined in terms of the *gradient operator*

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

This gradient operator operates not only on scalar-valued functions f, yielding the familiar gradient

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k},$$

but it also acts on vector fields in two different ways. If

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

is a smooth vector field, its *divergence* is the function

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z},$$

while its *curl* is the vector field

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} \\
&= \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ Q & R \end{vmatrix} \mathbf{i} + \begin{vmatrix} \partial/\partial z & \partial/\partial x \\ R & P \end{vmatrix} \mathbf{j} + \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ P & Q \end{vmatrix} \mathbf{k} \\
&= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}.
\end{aligned}$$

For example, if

$$\mathbf{F} = xy\mathbf{i} + x^2\mathbf{j} + z^2\mathbf{k},$$

then

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial z}(z^2) = y + 2z,$$

while

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & x^2 & z^2 \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(x^2) \right) \mathbf{i} + \left(\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(z^2) \right) \mathbf{j} + \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy) \right) \mathbf{k} \\
&= (2x + y)\mathbf{k}.
\end{aligned}$$

The geometrical and physical interpretations of the divergence and the curl come from the divergence theorem and Stokes's theorem.

The Divergence Theorem. *Let D be a region in (x, y, z)-space which is bounded by a piecewise smooth surface $\mathbf{S}$. Let $\mathbf{N}$ be the outward-pointing unit normal to $\mathbf{S}$. If $\mathbf{F}(x, y, z)$ is a vector field which is smooth on D and its boundary, then*

$$\iiint_D (\nabla \cdot \mathbf{F}) dxdydz = \iint_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N} dA.$$

The divergence theorem can often be used to reduce a complicated surface integral to a simpler volume integral, or a complicated volume integral to a simpler surface integral.

Example. The region

$$D = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$$

is bounded by the unit sphere S defined by the equation

$$x^2 + y^2 + z^2 = 1.$$

Suppose that we want to calculate the flux integral

$$\iint_S \mathbf{F}\cdot\mathbf{N}dA, \quad \text{where} \quad \mathbf{F} = 3x\mathbf{i} + e^z\mathbf{j} + x\cos y\mathbf{k}.$$

It would be hard to calculate this flux integral directly. However,

$$\nabla\cdot\mathbf{F} = \cdots = 3,$$

so using the fact that the volume enclosed by the sphere of radius r is $(4/3)\pi r^3$, we see that

$$\iint_S \mathbf{F}\cdot\mathbf{N}dA = \iiint_D 3dxdydz = 3(\text{Volume of } D) = 3\left(\frac{4}{3}\pi\right) = 4\pi.$$

The continuity equation from fluid mechanics. The reason the divergence theorem is so important is that it can be used to derive many of the important partial differential equations of mathematical physics. For example, we can use it to derive the *equation of continuity* from fluid mechanics.

Indeed, suppose that a fluid is flowing throughout (x,y,z)-space with velocity $\mathbf{V}(x,y,z,t)$ and density $\rho(x,y,z,t)$. If we represent the fluid flow by the vector field

$$\mathbf{F}(x,y,z,t) = \rho(x,y,z,t)\mathbf{V}(x,y,z,t),$$

then the surface integral

$$\iint_{\mathbf{S}} \mathbf{F}\cdot\mathbf{N}dA$$

represents the rate at which the fluid is flowing accross $\mathbf{S}$ in the direction of $\mathbf{N}$.

Let us assume that no fluid is being created or destroyed. Then the rate of change of the mass of fluid within D is given by two expressions,

$$\iiint_D \frac{\partial\rho}{\partial t}(x,y,z,t)dxdydz$$

and

$$-\iint_{\mathbf{S}} \mathbf{F}\cdot\mathbf{N}dA,$$

the latter being the rate at which fluid is crossing the boundary of D. It follows from the divergence theorem that the second of these expressions equals

$$-\iiint_D \nabla\cdot\mathbf{F}(x,y,z,t)dxdydz.$$

Thus

$$\iiint_D \frac{\partial\rho}{\partial t}dxdydz = -\iiint_D \nabla\cdot\mathbf{F}dxdydz.$$

Since this equation must hold for *every* region D in (x,y,z)-space, we conclude that the integrands must be equal,

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{F} = -\nabla\cdot(\rho\mathbf{V}).$$

Thus we obtain the equation of continuity,

$$\frac{\partial \rho}{\partial t}(x, y, z, t) + \nabla \cdot (\rho \mathbf{V})(x, y, z, t) = 0. \tag{1.8}$$

This derivation illustrates one of the main applications of the divergence theorem, to the derivation of equations used in mathematical models; the equation of continuity is only one of the important *Euler equations* or the *Navier-Stokes equations* which are used to model a perfect fluid in fluid mechanics.

Fluid mechanics gives an interpretation of divergence: We can imagine that a smooth vector field $\mathbf{F}(x, y, z)$ is of the form

$$\mathbf{F}(x, y, z) = \rho(x, y, z)\mathbf{V}(x, y, z),$$

where ρ is the density and $\mathbf{V}$ is the velocity of smooth steady-state fluid (but we do not assume that no fluid is being created or destroyed). Then $\nabla \cdot \mathbf{F}$ represents the rate at which fluid is being created (if positive) or destroyed (if negative) per unit volume.

Exercises:

1.7.1.a. Find the divergence of the vector field

$$\mathbf{F}(x, y, z) = \log(y^2 + z^2 + 1)\mathbf{i} + y\mathbf{j} + (\sin x \cos y)\mathbf{k}.$$

b. Use the divergence theorem to evaluate the flux integral

$$\iint_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N} dA,$$

where $\mathbf{S}$ is the boundary of the cube

$$-1 \leq x \leq 1, -1 \leq y \leq 1, -1 \leq z \leq 1,$$

and $\mathbf{N}$ is the outward-pointing unit normal.

1.7.2.a. Find the divergence of the vector field

$$\mathbf{F}(x, y, z) = e^{-y^2}\mathbf{i} + xz\mathbf{j} + z^3\mathbf{k}.$$

b. Use the divergence theorem to evaluate the flux integral

$$\iint_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N} dA,$$

where $\mathbf{S}$ is the boundary of the cylindrical region

$$x^2 + y^2 \leq 1, \qquad 0 \leq z \leq 1,$$

and $\mathbf{N}$ is the outward-pointing unit normal.

1.7.3.a. Find the divergence of the vector field

$$\mathbf{F}(x,y,z) = (e^{y^2+z^2})\mathbf{i} + 7y\mathbf{j} + (z-5)\mathbf{k}.$$

b. Use the divergence theorem and the change of variables formula for a multiple integral to evaluate the flux integral

$$\iint_S \mathbf{F} \cdot \mathbf{N} dA,$$

where S is the boundary of the region

$$D = \left\{(x,y,z) \in R^3 : (2x+z)^2 + (y+z)^2 + z^2 \leq 1\right\}$$

and $\mathbf{N}$ is the outward-pointing unit normal. (You may use the fact that the volume bounded by the unit sphere is $(4/3)\pi$.)

1.8 Stokes's Theorem

The other major integral theorem from vector calculus is:

Stokes's Theorem. *If* $\mathbf{S}$ *is an oriented smooth surface in* R^3 *bounded by a piecewise smooth curve* $\partial\mathbf{S}$, *and* $\mathbf{F}$ *is a smooth vector field on* R^3, *then*

$$\iint_{\mathbf{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{N} dA = \int_{\partial\mathbf{S}} \mathbf{F} \cdot \mathbf{T} ds,$$

where $\mathbf{N}$ *is the unit normal chosen by the orientation and* $\mathbf{T}$ *is the unit tangent to* $\partial\mathbf{S}$ *chosen so that* $\mathbf{N} \times \mathbf{T}$ *points into* $\mathbf{S}$.

Stokes's theorem can be used in two directions, to reduce a complicated line integral to a simpler surface integral or a complicated surface integral to a simpler line integral.

Example. Suppose that S is the part of the sphere $x^2 + y^2 + z^2 = 1$ lying in the first octant $x \geq 0$, $y \geq 0$, $z \geq 0$, with the orientation determined by the upward-pointing unit normal $\mathbf{N}$. In this case, the boundary of S is the curve $\mathbf{C} = \partial S$, and it is traversed in the counterclockwise direction as viewed from above. We can think of $\mathbf{C}$ as consisting of three segments of circles

$$\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2 + \mathbf{C}_3,$$

where $\mathbf{C}_1$ goes from $(1,0,0)$ to $(0,1,0)$, $\mathbf{C}_2$ goes from $(0,1,0)$ to $(0,0,1)$ and $\mathbf{C}_2$ goes from $(0,0,1)$ to $(1,0,0)$. Stokes' Theorem states that if $\mathbf{F}$ is any smooth vector field on R^3, then

$$\begin{aligned} \int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{N} dA = \int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{T} ds &= \int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{x} \\ &= \int_{\mathbf{C}_1} \mathbf{F} \cdot \mathbf{dx} + \int_{\mathbf{C}_2} \mathbf{F} \cdot \mathbf{dx} + \int_{\mathbf{C}_3} \mathbf{F} \cdot \mathbf{dx}. \end{aligned}$$

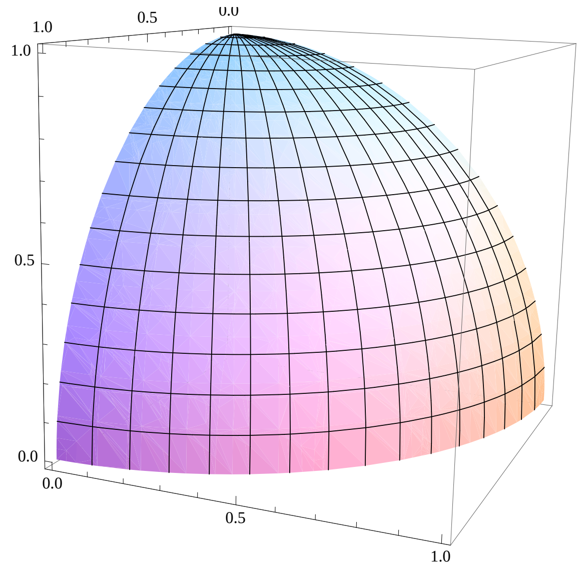

Figure 1.9: The part of the unit sphere which lies in the first octant.

If we want to carry out the line integral, we can give $\mathbf{C}_1$, $\mathbf{C}_2$ and $\mathbf{C}_2$ the parametrizations

$$\mathbf{x}_1 : [0, \pi/2] \to R^3, \qquad \mathbf{x}_1(t) = (\cos t, \sin t, 0),$$
$$\mathbf{x}_2 : [0, \pi/2] \to R^3, \qquad \mathbf{x}_2(t) = (0, \cos t, \sin t),$$
$$\mathbf{x}_3 : [0, \pi/2] \to R^3, \qquad \mathbf{x}_3(t) = (\sin t, 0, \cos t),$$

Suppose that

$$\mathbf{F}(x, y, z) = \sin(x^2 + 1)\mathbf{i} + z^2\mathbf{j} + \cos(e^z)\mathbf{k}.$$

We ask: what is

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{dx}?$$

Direct calculation of the line integrals in this case would be difficult. On the other hand,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin(x^2+1) & z^2 & \cos(e^z) \end{vmatrix} = \cdots = -2z\mathbf{i}.$$

and Stokes' Theorem says that

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{dx} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N} dA.$$

Since the curl of $\mathbf{F}$ is relatively simple, the surface integral will be easier to calculate.

To parametrize the part of the sphere lying in the first octant, it is natural to use spherical coordinates:

$$\begin{cases} x = \rho \sin\phi \cos\theta, \\ y = \rho \sin\phi \sin\theta, \\ z = \rho \cos\phi, \end{cases} \qquad \begin{cases} x = \sin u \cos v, \\ y = \sin u \sin v, \\ z = \cos u, \end{cases}$$

where $0 \le u \le \pi/2$, $0 \le v \le \pi/2$. In other words, we use the parametrization $\mathbf{x} : D \to R^3$, where

$$D = \{(u,v) : 0 \le u \le \pi/2,\ 0 \le v \le \pi/2\}, \quad \mathbf{x}(u,v) = \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix}.$$

$$\frac{\partial \mathbf{x}}{\partial u} = \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ -\sin u \end{pmatrix}, \quad \frac{\partial \mathbf{x}}{\partial v} = \begin{pmatrix} -\sin u \sin v \\ \sin u \cos v \\ 0 \end{pmatrix} = \sin u \begin{pmatrix} -\sin v \\ \cos v \\ 0 \end{pmatrix},$$

$$\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} = \sin u \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos u \cos v & \cos u \sin v & -\sin u \\ -\sin v & \cos v & 0 \end{vmatrix} = \sin u \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix},$$

$$\mathbf{N} dA = \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} du\ dv = \sin u \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix} du\ dv.$$

$$\nabla \times \mathbf{F} = -2z\mathbf{i} = \begin{pmatrix} -2\cos u \\ 0 \\ 0 \end{pmatrix}, \quad \nabla \times \mathbf{F} \cdot \mathbf{N} dA = -2\sin^2 u \cos u \cos v\ dudv.$$

Thus the calculation yields

$$\begin{aligned} \int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{dx} = \iint_S \mathbf{F} \cdot \mathbf{N} dA \\ &= -\int_0^{\pi/2} \int_0^{\pi/2} 2\sin^2 u \cos u \cos v\ dudv \\ &= -\left[\int_0^{\pi/2} 2\sin^2 u \cos u\ du\right]\left[\int_0^{\pi/2} \cos v\ dv\right] = -\frac{2}{3}. \end{aligned}$$

Interpretation of curl: Stokes's theorem gives rise to a geometric interpretation of curl. Indeed, let (x_0, y_0, z_0) be a given point in R^3, $\mathbf{N}$ a unit-length vector located at (x_0, y_0, z_0), Π the plane through (x_0, y_0, z_0) which is perpendicular to $\mathbf{N}$. Let D_ϵ be the disk in Π of radius ϵ centered at (x_0, y_0, z_0), ∂D_ϵ the circle in Π of radius ϵ centered at (x_0, y_0, z_0). Then

$$(\nabla \times \mathbf{F})(x_0, y_0, z_0) \cdot \mathbf{N} = \lim_{\epsilon \to 0} \frac{\int\int_{D_\epsilon} (\nabla \times \mathbf{F}) \cdot \mathbf{N} dA}{\text{Area of } D_\epsilon}$$

$$= \lim_{\epsilon \to 0} \frac{\int_{\partial D_\epsilon} \mathbf{F} \cdot \mathbf{T} ds}{\text{Area of } D_\epsilon} = \lim_{\epsilon \to 0} \frac{\text{Rate of circulation of } \mathbf{F} \text{ about } \partial D_\epsilon}{\text{Area of } D_\epsilon}.$$

Thus $(\nabla \times \mathbf{F})(x_0, y_0, z_0)$ measures the rate of circulation of the fluid flow represented by $\mathbf{F}$ near (x_0, y_0, z_0).

Digression: Maxwell's Equations. In 1864, James Clerk Maxwell found a set of partial differential equations which were so powerful that they could be used to derive the entire theory of electricity and magnetism, yet so simple that they could be written on the back of a postcard. Maxwell's equations are formulated in terms of divergence and curl. To utilize Maxwell's equations in applications requires the divergence theorem and Stokes's theorem.

This is not the place to derive Maxwell's equations; this is done in upper division courses in electricity and magnetism[1]. However, we would like to mention that one can derive consequences from Maxwell's equations by means of the divergence theorem and Stokes's theorem. Maxwell's equations are usually stated in terms of

$$\text{the charge density} \quad \rho(x, y, z, t),$$

$$\text{the current density} \quad \mathbf{J}(x, y, z, t),$$

$$\text{the electric field} \quad \mathbf{E}(x, y, z, t),$$

and

$$\text{the magnetic field} \quad \mathbf{B}(x, y, z, t).$$

In terms of these variables, Maxwell's equations are a set of four partial differential equations which make use of divergence and curl:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \cdot \mathbf{B} = 0, \tag{1.9}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}. \tag{1.10}$$

Here c is the speed of light and ϵ_0 and μ_0 are constants. (The exact form of the equations depend on units used and other conventions.) In the case where all functions are independent of time, these equations simplify to

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = 0, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Often, the charge density ρ and the current density $\mathbf{J}$ are given, and one determines the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ by solving Maxwell's equations. The electric and magnetic fields can then determine the force acting on a test particle of charge Q in accordance with the formula

$$\mathbf{F} = Q\mathbf{E} + Q(\mathbf{v} \times \mathbf{B}),$$

[1] A derivation of Maxwell's equations is presented in Lorrain and Corson, *Electromagnetic fields and waves*, Second equation, Freeman, San Francisco, 1970.

where $\mathbf{v}$ is the velocity of the particle.

The divergence theorem applied to the Maxwell equation $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ yields *Gauss's law*: If D is a region in three-space bounded by a smooth surface ∂D, then

$$\iint_{\partial D} \mathbf{E} \cdot \mathbf{N} dA = \iiint_D \nabla \cdot \mathbf{E} dx dy dz = \frac{1}{\epsilon_0} \iiint_D \rho dx dy dz = \frac{Q}{\epsilon_0},$$

where Q denotes the total charge within D. In other words, the total flux of $\mathbf{E}$ outward through the surface ∂D equals $1/\epsilon_0$ times the total charge within D.

On the other hand, Stokes's theorem applied to the time-independent Maxwell equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

yields *Ampère's law*: If S is a surface bounded by a smooth curve ∂S, then

$$\int_{\partial S} \mathbf{B} \cdot \mathbf{T} ds = \iint_S \nabla \times \mathbf{B} \cdot \mathbf{N} dA = \mu_0 \iint_S \mathbf{J} \cdot \mathbf{N} dA.$$

In other words, the integral of the tangential component of $\mathbf{B}$ around the curve ∂S equals μ_0 times the total current flowing through S.

Exercises:

1.8.1. Let $\mathbf{C}$ be the boundary of the part of the ellispsoid $x^2 + y^2 + 4z^2 = 4$ which lies in the first octant $x \geq 0, y \geq 0, z \geq 0$, oriented to be traversed in the counterclockwise manner as viewed from above. Use Stokes's theorem to evaluate the line integral

$$\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{x} = \int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{T} ds,$$

where $\mathbf{F}(x, y, z) = (\sin(3x), 3x, e^{-z^2})$.

1.8.2. Suppose that S is the part of the plane $x + y + z = 1$ which lies in the octant defined by $x \geq 0$, $y \geq 0$ and $z \geq 0$, and that $\mathbf{C}$ is the boundary of S oriented counterclockwise as viewed from above. Use Stokes's theorem to evaluate the line integral

$$\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{x} = \int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{T} ds,$$

where $\mathbf{F}(x, y, z) = (\cos(x^2 + 1), e^{-y^2}, x + 2y)$.

1.8.3.a. Let S be the part of the paraboloid $z = x^2 + y^2$ which satisfies the inequalities $0 \leq x \leq 1$, $0 \leq y \leq 1$. Give a parametrization for S.

b. Use Stokes's Theorem to evaluate the line integral

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \int_{\partial S} \mathbf{F} \cdot \mathbf{T} ds, \qquad \text{where} \qquad \mathbf{F} = (-y^2, e^y, \cos(\log(z^2 + 5)))$$

and ∂S is the boundary of S, oriented counterclockwise as viewed from above.

Chapter 2

Power Series

2.1 Convergence of infinite sequences

A *sequence* of real numbers is simply a function which assigns to each natural number n a real number s_n. (In other words, a sequence of real numbers is simply a function from the set N of natural numbers to the set R of real numbers.) We sometimes use the notation (s_n) for the sequence which assigns s_n to the natural number n. For example, we can define a sequence $n \mapsto s_n$ by

$$s_n = \frac{1}{n}, \quad n = 1, 2, 3, 4, \ldots .$$

The key definition from the theory of infinite sequences is:

Definition. A sequence (s_N) of real numbers is said to *converge* to a real number s if for every prescribed error $\epsilon \in R$ with $\epsilon > 0$, there is a natural number N such that

$$n > N \quad \Rightarrow \quad |s_n - s| < \epsilon.$$

In this case, we write $s = \lim s_n$. A sequence (s_n) of real numbers which does not converge to a real number is said to *diverge*.

In rough terms, the definition says that a sequence (s_n) converges to a real number s if and only if s_n gets closer and closer to s as $n \to \infty$.

Example 1. We claim that the sequence (s_n) defined by $s_n = 1/n$ converges to 0. Given an error $\epsilon > 0$, there exists a natural number N such that $N > 1/\epsilon$ and thus $1/N < \epsilon$. It follows that

$$n > N \quad \Rightarrow \quad 0 < \frac{1}{n} < \frac{1}{N} \quad \Rightarrow \quad |s_n - 0| = \left| \frac{1}{n} - 0 \right| < \epsilon.$$

Using the same technique, you could show that the sequence (s_n) defined by $s_n = a/n$ converges to 0, whenever a is a real number.

Example 2. On the other hand, the sequence (s_n) defined by $s_n = 1 + (-1)^n$ diverges. Indeed, suppose that this sequence (s_n) were to converge to s. We could then take $\epsilon = 1$, and there would exist a natural number N such that

$$n > N \quad \Rightarrow \quad |s_n - s| < \frac{1}{2}.$$

But then if $n > N$ and n is even, we would have $s_n = 2$ and $s_{n+1} = 0$. Hence,

$$2 = |s_n - s_{n+1}| \leq |s_n - s| + |s - s_{n+1}| < \frac{1}{2} + \frac{1}{2} = 1,$$

a contradiction.

Example 3. Suppose we define the sequence (s_n) by

$$s_n = \frac{2n+3}{n+5},$$

which we can rewrite as

$$s_n = \frac{2 + 3/n}{1 + 5/n}.$$

By Example 1, we see that as $n \to \infty$, $(3/n)$ and $(5/n)$ converge to zero. Thus we expect (s_n) to converge to $2/1 = 2$. To construct an error estimate, we would investigate the inequality

$$\left| \frac{2n+3}{n+5} - 2 \right| < \epsilon \quad \text{or} \quad \left| \frac{2n+3-2(n+5)}{n+5} \right| < \epsilon.$$

We can rewrite this as

$$\left| \frac{-7}{n+5} \right| < \epsilon \quad \text{or} \quad \left| \frac{7}{\epsilon} \right| < n+5.$$

We can choose the natural number N so that $N > 7/\epsilon$. Then

$$\begin{aligned} n > N \quad \Rightarrow \quad n > \frac{7}{\epsilon} \quad \Rightarrow \quad & \frac{7}{n} < \epsilon \\ \Rightarrow \quad & \left| \frac{2n+3}{n+5} - 2 \right| < \epsilon \quad \Rightarrow \quad |s_n - 2| < \epsilon. \end{aligned}$$

Example 4. An important example from first year calculus is the sequence (s_n) defined by

$$s_n = \left(1 + \frac{k}{n}\right)^n.$$

for example, k could represent the interest rate on a bank deposit (with $k = .1$ corresponding to 10% interest). If the bank does no compounding, each dollar in the account will grow to

$$s_1 = (1 + k)$$

dollars at the end of one year. But if the bank compounds four times a year, once every three months, the amount at the end of one year is

$$s_4 = \left(1 + \frac{k}{4}\right)^4$$

which is slightly larger than $1 + k$ because of compounding. Similarly,

$$s_{12} = \left(1 + \frac{k}{12}\right)^{12}$$

is the amount in the account if interest is compounded monthly, which is even larger. The limit of the sequence (s_n) represents the amount each dollar in the account will grow to under instantaneous compounding, and that is a well-known limit which you should have studied in calculus,

$$\lim_{n\to\infty}(s_n) = \left(1 + \frac{k}{n}\right)^n = e^k.$$

In this course, we will use limits of infinite sequences to define convergence of two important types of infinite sums, power series and Fourier series, which are useful for solving ordinary and partial differential equations.

Exercise:

2.1.1. Determine which of the following infinite sequences converge, and determine the limit in the case of convergence:

a. $s_n = 1/n + 5$, b. $s_n = 1/n^2$,

c. $s_n = (n + 3)/(2n + 1)$, d. $s_n = \cos(n\pi)$.

2.2 Convergence of power series

Functions are often represented efficiently by means of infinite sums, also known as infinite series. Examples you may have seen in calculus include the exponential function

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n, \tag{2.1}$$

as well as the trigonometric functions

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots = \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!}x^{2k}$$

and

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots = \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)!}x^{2k+1}.$$

We show how to derive these series representations from Taylor's Theorem in §2.3. For now, it is important to realize that an infinite series of this type is called a power series. To be precise, a *power series* centered at x_0 is an infinite sum of the form

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where the a_n's are constants. In advanced treatments of calculus, the above power series representations are often used to define the exponential and trigonometric functions.

Power series can also be used to construct tables of values for these functions. For example, using a calculator or PC with suitable software installed (such as Mathematica), we could calculate

$$1 + 1 + \frac{1}{2!}1^2 = \sum_{n=0}^{2} \frac{1}{n!}1^n = 2.5,$$

$$1 + 1 + \frac{1}{2!}1^2 + \frac{1}{3!}1^3 + \frac{1}{4!}1^4 = \sum_{n=0}^{4} \frac{1}{n!}1^n = 2.70833,$$

$$\sum_{n=0}^{8} \frac{1}{n!}1^n = 2.71806, \qquad \sum_{n=0}^{12} \frac{1}{n!}1^n = 2.71828, \qquad \ldots$$

As the number of terms increases, the sum approaches the familiar value of the exponential function e^x at $x = 1$.

For a power series to be useful, the infinite sum must actually add up to a finite number, as in this example, for at least some values of the variable x. We let s_N denote the sum of the first $N + 1$ terms in the power series,

$$s_N = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_N(x - x_0)^N = \sum_{n=0}^{N} a_n (x - x_0)^n,$$

and say that the power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

converges if and only if $\lim_{N\to\infty} s_N$ exists, that is, if and only if the finite sum s_N gets closer and closer to some (finite) number as $N \to \infty$. Otherwise, we say that the power series *diverges*.

Let us consider, for example, one of the most important power series of applied mathematics, the *geometric series*

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n. \tag{2.2}$$

In this case we have

$$s_N = 1 + x + x^2 + x^3 + \cdots + x^N, \qquad xs_N = x + x^2 + x^3 + x^4 \cdots + x^{N+1},$$

$$s_N - xs_N = 1 - x^{N+1}, \qquad s_N = \frac{1 - x^{N+1}}{1 - x}.$$

If $|x| < 1$, then x^{N+1} gets smaller and smaller as N approaches infinity, and hence

$$\lim_{N \to \infty} x^{N+1} = 0.$$

Substituting into the expression for s_N, we find that

$$\lim_{N \to \infty} s_N = \frac{1}{1 - x}.$$

Thus if $|x| < 1$, we say that the geometric series converges, and write

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}.$$

On the other hand, if $|x| > 1$, then x^{N+1} gets larger and larger as N approaches infinity, so $\lim_{N \to \infty} x^{N+1}$ does not exist as a finite number, and neither does $\lim_{N \to \infty} s_N$. In this case, we say that the geometric series *diverges*. In summary, the geometric series

$$\sum_{n=0}^{\infty} x^n \qquad \text{converges to} \quad \frac{1}{1 - x} \qquad \text{when} \quad |x| < 1,$$

and diverges when $|x| > 1$. One can check directly that the sequence diverges for $x = \pm 1$, so the geometric series (2.2) converges if and only if $x \in (-1, 1)$.

This behaviour, convergence for $|x| <$ some number, and divergences for $|x| >$ that number, is one of the key features of power series:

Theorem. *For any power series*

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

there exists R, which is a nonnegative real number or ∞, such that

1. *the power series converges when $|x - x_0| < R$,*
2. *and the power series diverges when $|x - x_0| > R$.*

We call R the *radius of convergence*. A proof of this theorem is given in more advanced courses on real or complex analysis.[1]

[1] Good references for the theory behind convergence of power series are Edward D. Gaughan, *Introduction to analysis*, Brooks/Cole Publishing Company, Pacific Grove, 1998 and Walter Rudin, *Principles of mathematical analysis*, third edition, McGraw-Hill, New York, 1976.

We have seen that the geometric series

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

has radius of convergence $R = 1$. More generally, if b is a positive constant, the power series

$$1 + \frac{x}{b} + \left(\frac{x}{b}\right)^2 + \left(\frac{x}{b}\right)^3 + \cdots = \sum_{n=0}^{\infty} \left(\frac{x}{b}\right)^n \tag{2.3}$$

has radius of convergence b. To see this, we make the substitution $y = x/b$, and the power series becomes $\sum_{n=0}^{\infty} y^n$, which we already know converges for $|y| < 1$ and diverges for $|y| > 1$. But

$$|y| < 1 \quad \Leftrightarrow \quad \left|\frac{x}{b}\right| < 1 \quad \Leftrightarrow \quad |x| < b,$$

$$|y| \geq 1 \quad \Leftrightarrow \quad \left|\frac{x}{b}\right| \geq 1 \quad \Leftrightarrow \quad |x| \geq b.$$

Thus for $|x| < b$ the power series (2.3) converges to

$$\frac{1}{1-y} = \frac{1}{1-(x/b)} = \frac{b}{b-x},$$

while for $|x| \geq b$, it diverges. Thus when b is a positive number, the power series (2.3) converges if and only if $x \in (-b, b)$.

Here is a simple criterion that often enables one to determine the radius of convergence of a power series:

Ratio Test. The radius of convergence of the power series

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

is given by the formula

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|},$$

so long as this limit exists.

Let us check that the ratio test gives the right answer for the radius of convergence of the power series (2.3). In this case, we have

$$a_n = \frac{1}{b^n}, \quad \text{so} \quad \frac{|a_n|}{|a_{n+1}|} = \frac{1/b^n}{1/b^{n+1}} = \frac{b^{n+1}}{b^n} = b,$$

and the formula from the ratio test tells us that the radius of convergence is $R = b$, in agreement with our earlier determination.

In the case of the power series for e^x,

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n,$$

in which $a_n = 1/n!$, we have

$$\frac{|a_n|}{|a_{n+1}|} = \frac{1/n!}{1/(n+1)!} = \frac{(n+1)!}{n!} = n+1,$$

and hence

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} (n+1) = \infty,$$

so the radius of convergence is infinity. In this case the power series converges for all x. This shows that we could use the power series expansion for e^x to calculate e^x for any choice of x.

On the other hand, in the case of the power series

$$\sum_{n=0}^{\infty} n! x^n,$$

in which $a_n = n!$, we have

$$\frac{|a_n|}{|a_{n+1}|} = \frac{n!}{(n+1)!} = \frac{1}{n+1}, \qquad R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \left(\frac{1}{n+1} \right) = 0.$$

In this case, the radius of convergence is zero, and the power series does not converge for any nonzero x.

The ratio test doesn't always work because the limit may not exist, but sometimes one can use it in conjunction with the

Comparison Test. Suppose that the power series

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n, \qquad \sum_{n=0}^{\infty} b_n (x-x_0)^n$$

have radius of convergence R_1 and R_2 respectively. If $|a_n| \leq |b_n|$ for all n, then $R_1 \geq R_2$. If $|a_n| \geq |b_n|$ for all n, then $R_1 \leq R_2$.

In short, power series with smaller coefficients have larger radius of convergence.

Consider for example the power series expansion for $\cos x$,

$$1 + 0x - \frac{1}{2!}x^2 + 0x^3 + \frac{1}{4!}x^4 - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} x^{2k}.$$

In this case the coefficient a_n is zero when n is odd, while $a_n = \pm 1/n!$ when n is even. In either case, we have $|a_n| \leq 1/n!$, so we can compare with the power series

$$1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

which represents e^x and has infinite radius of convergence. It follows from the comparison test that the radius of convergence of

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!}x^{2k}$$

must be at least large as that of the power series for e^x, and hence must also be infinite.

Power series with positive radius of convergence are so important that there is a special term for describing functions which can be represented by such power series. A function $f(x)$ is said to be *real analytic* at x_0 if there is a power series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

about x_0 with positive radius of convergence R such that

$$f(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n, \qquad \text{for} \qquad |x-x_0|<R.$$

For example, the functions e^x is real analytic at any x_0. To see this, we utilize the law of exponents to write $e^x = e^{x_0}e^{x-x_0}$ and apply (2.1) with x replaced by $x-x_0$:

$$e^x = e^{x_0}\sum_{n=0}^{\infty}\frac{1}{n!}(x-x_0)^n = \sum_{n=0}^{\infty} a_n(x-x_0)^n, \qquad \text{where} \qquad a_n=\frac{e^{x_0}}{n!}.$$

This is a power series expansion of e^x about x_0 with infinite radius of convergence. Similarly, the monomial function $f(x)=x^n$ is real analytic at x_0 because

$$x^n=(x-x_0+x_0)^n=\sum_{i=0}^{n}\frac{n!}{i!(n-i)!}x_0^{n-i}(x-x_0)^i$$

by the binomial theorem, a power series about x_0 in which all but finitely many of the coefficients are zero.

A key feature of analytic functions is that they have continuous derivatives of all orders.

In more advanced courses, one studies conditions under which functions are real analytic. For our purposes, it is sufficient to be aware of the following facts: The sum and product of real analytic functions is real analytic. It follows from this that any polynomial

$$P(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$$

is analytic at any x_0. The quotient of two polynomials with no common factors, $P(x)/Q(x)$, is analytic at x_0 if and only if x_0 is not a zero of the denominator

$Q(x)$. Thus for example, $1/(x-1)$ is analytic whenever $x_0 \neq 1$, but fails to be analytic at $x_0 = 1$.

Exercises:

2.2.1. Use the ratio test to find the radius of convergence of the following power series:

$$\text{a.} \quad \sum_{n=0}^{\infty} (-1)^n x^n, \qquad \text{b.} \quad \sum_{n=0}^{\infty} \frac{1}{n+1} x^n,$$

$$\text{c.} \quad \sum_{n=0}^{\infty} \frac{3}{n+1} (x-2)^n, \qquad \text{d.} \quad \sum_{n=0}^{\infty} \frac{1}{2^n} (x-\pi)^n,$$

$$\text{e.} \quad \sum_{n=0}^{\infty} (7x-14)^n, \qquad \text{f.} \quad \sum_{n=0}^{\infty} \frac{1}{n!} (3x-6)^n.$$

2.2.2. Use the comparison test to find an estimate for the radius of convergence of each of the following power series:

$$\text{a.} \quad \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k}, \qquad \text{b.} \quad \sum_{k=0}^{\infty} (-1)^k x^{2k},$$

$$\text{c.} \quad \sum_{k=0}^{\infty} \frac{1}{2k+1} (x-4)^{2k} \qquad \text{d.} \quad \sum_{k=0}^{\infty} \frac{1}{2^{2k}} (x-\pi)^{2k}.$$

2.2.3. Use the comparison test and the ratio test to find the radius of convergence of the power series

$$\sum_{m=0}^{\infty} (-1)^m \frac{1}{(m!)^2} \left(\frac{x}{2}\right)^{2m}.$$

2.2.4. Determine the values of x_0 at which the following functions fail to be real analytic:

$$\text{a.} \quad f(x) = \frac{1}{x-4}, \qquad \text{b.} \quad g(x) = \frac{x}{x^2-1},$$

$$\text{c.} \quad h(x) = \frac{4}{x^4 - 3x^2 + 2}, \qquad \text{d.} \quad \phi(x) = \frac{1}{x^3 - 5x^2 + 6x}$$

2.3 Taylor series

How does one determine the power series expansion of e^x given by (2.1)? The simplest approach is to make use of one of the key theorems from calculus, Taylor's Theorem. Taylor's Theorem simply says that a function with continuous partial derivatives up to order $n+1$ near a given point x_0 can be well approximated by a polynomial of degree n near x_0.

Taylor's Theorem. *If a function $f : R \to R$ has continuous partial derivatives up to order $n+1$, then*

$$f(x_0+h) = f(x_0)+f'(x_0)h+\frac{1}{2!}f''(x_0)h^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)h^n+R_n(x_0,h), \quad (2.4)$$

with the remainder term $R_k(x_0,h)$ satisfying the estimate

$$|R_n(x_0,h)| \leq \frac{1}{k!}M_{n+1}|h|^{n+1}, \quad (2.5)$$

whenever M_{n+1} is greater than the maximum value of the derivative $f^{(n+1)}(x)$ for $|x-x_0| \leq |h|$.

If we write $x = x_0 + h$, we can also write the formula for Taylor's Theorem as

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!}f''(x_0)(x-x_0)^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n + R_n(x_0, x-x_0), \quad (2.6)$$

or use the "sigma notation" and write it as

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!}f^{(k)}(x_0)(x-x_0)^k + R_n(x_0, x-x_0).$$

The polynomial

$$\begin{aligned} p_n(x) &\\ = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!}f''(x_0)(x-x_0)^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n &\\ = \sum_{k=0}^{n} \frac{1}{k!}f^{(k)}(x_0)(x-x_0)^k & \end{aligned}$$

is called the n-th order *Taylor polynomial* for f at x_0, or the n-th order *Maclaurin polynomial* for f if $x_0 = 0$. The estimate (2.5) tells how closely the Taylor polynomial approximates the function f near x_0.

If f has infinitely many derivatives, and we have estimates on these derivaties, we can often show that

$$\lim_{n\to\infty} |R_n(x_0,h)| = 0$$

for all h such that $|h| \leq R$ for some $R > 0$. When this happens we can let $n \to \infty$ in (2.6), thereby obtaining a power series representation for f,

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!}f^{(k)}(x_0)(x-x_0)^k,$$

which is called the *Taylor series* for f centered at x_0 or the *Maclaurin series* for f when $x_0 = 0$. When f has a Taylor series at x_0 with positive radius on convergence, it is real analytic at x_0.

Example 1. Suppose that $f(x) = e^x$. Then

$$f^{(k)}(x) = e^x, \quad f^{(k)}(0) = 1, \quad \text{for all } k.$$

Hence Taylor's Theorem with $x_0 = 0$ yields

$$e^x = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + R_n(0,h),$$

where

$$|R_n(0,h)| \leq \frac{1}{n!}|h|^{n+1}.$$

In this case, the factorial wins out over the power, and for any fixed value of h,

$$\lim |R_n(0,h)| = 0.$$

Thus Taylor's Theorem yields the power series representation for the exponential function,

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}x^k.$$

The n-th Taylor polynomial in this case is

$$p_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n.$$

Example 2. Now consider the function

$$f(x) = \log(x) \quad \text{And set} \quad x_0 = 1.$$

(Note that in this course, the logarithm will always be the natural logarithm which has base e.) Then

$$f(1) = \log 1 = 0, \quad f'(x) = \frac{1}{x} \quad \text{so} \quad f'(1) = 1.$$

$$f''(x) = -\frac{1}{x^2} \quad \text{so} \quad f''(1) = -1, \quad f^{(3)}(x) = \frac{2!}{x^3} \quad \text{so} \quad f^{(3)}(1) = 2!$$

and in general

$$f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{x^n}, \quad \text{so} \quad f^{(n)}(1) = (-1)^{n+1}(n-1)!.$$

Thus assuming it converges, the Taylor series expansion is

$$\log(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots + \frac{(-1)^{n+1}}{n}(x-1)^n + \cdots.$$

It turns out that the radius of convergence of the Taylor series is one; can you show this?

Idea behind the proof of Taylor'sTheorem: To derive Taylor's Theorem, one starts with the Fundamental Theorem of Calculus, that integration is the inverse to differentiation, and uses it to show that

$$f(x) - f(x_0) = f(x_0 + h) - f(x_0) = \int_0^1 f'(x_0 + th) h dt.$$

We can write this as

$$f(x_0 + h) = f(x_0) + \int_0^1 f'(x_0 + th) h dt, \tag{2.7}$$

or as

$$f(x_0 + h) = f(x_0) + R_0(x_0, h), \quad \text{where} \quad R_0(x_0, h) = \int_0^1 f'(x_0 + th) h dt.$$

This is the zeroth order version of Taylor's theorem, with the remainder term $R_0(x_0, h)$ satisfying the inequality

$$|R_0(x_0, h)| \leq M_1 |h|,$$

where M_1 is the maximum value of $|f'(x)|$ when $|x - x_0| \leq |h|$.

We can next apply the Fundamental Theorem of Calculus to $f'(x_0 + th)$, yielding

$$f'(x_0 + th) = f'(x_0) + \int_0^1 f''(x_0 + sth) th ds,$$

and substitute into (2.7) to obtain

$$f(x_0 + h) = f(x_0) + \int_0^1 \left[f'(x_0) + \int_0^1 f''(x_0 + sth) th ds \right] h dt,$$

or

$$f(x_0 + h) = f(x_0) + f'(x_0) h + \int_0^1 \int_0^1 f''(x_0 + sth) t h^2 ds dt, \tag{2.8}$$

which can be written as

$$f(x_0 + h) = f(x_0) + f'(x_0) h + R_1(x_0, h),$$
$$\text{where} \quad R_1(x_0, h) = \int_0^1 \int_0^1 f''(x_0 + sth) t h^2 ds dt. \tag{2.9}$$

This time the error term $R_1(x_0, h)$ satisfies the inequality

$$|R_1(x_0, h)| \leq M_2 \int_0^1 \int_0^1 t ds dt |h|^2 \leq \frac{1}{2!} M_2 |h|^2,$$

where M_2 is the maximum value of $|f''(x)|$ when $|x - x_0| \leq |h|$.

What the estimate (2.9) says is that $f(x)$ is well approximated by the linear Taylor polynomial

$$p_1(x) = f(x_0) + f'(x_0)(x - x_0),$$

with the error being small as long as the second derivative of f is small.

Applying the Fundamental Theorem of Calculus to $f''(x_0 + sth)$, we would obtain the second order Taylor approximation,

$$f(x_0+h) = f(x_0)+f'(x_0)h+\frac{1}{2!}f''(x_0)h^2+\int_0^1\int_0^1\int_0^1 f'''(x_0+usth)st^2h^2dudsdt, \tag{2.10}$$

which can be written as

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{1}{2!}f''(x_0)h^2 + R_2(x_0, h),$$

where

$$|R_2(x_0, h)| \leq \frac{1}{3!}M_3|h|^3.$$

This shows that $f(x)$ is well approximated by the quadratic Taylor polynomial

$$p_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2,$$

with the error being small as long as the third derivative of f is small.

We can continue in the same fashion, applying the Fundamental Theorem of Calculus to higher and higher derivatives, to obtain higher and higher order Taylor approximations. To make this idea into a complete argument, one would use mathematical induction, as described in basic courses on mathematical proof.

Exercises:

2.3.1. Find the n-to order Maclaurin polynomial (or the n-th order Taylor polynomial centered at $x = 0$) for each of the following functions:

a. $f(x) = e^{-x}$, b. $f(x) = \log(x + 1)$,

c. $f(x) = 1/(x + 1)$, d. $f(x) = \cosh x = (1/2)(e^x + e^{-x})$.

2.3.2. Find the Taylor series for each of the following functions centered at the indicated point:

a. $f(x) = e^x$, $x = 1$, b. $f(x) = \log(x + 1)$, $x = 0$,

c. $f(x) = \cos x$, $x = \pi$, d. $f(x) = \cosh x$, $x = 0$.

2.4 Solving differential equations by means of power series

In the eighteenth and nineteenth century, mathematicians encountered many ordinary differential equations which they could not solve explicitly in terms of known functions such as e^x and $\cos x$. But they were able to find power series expansions of the solutions, which enabled them to determine tables of values for the solutions for various choices of initial conditions. Then most frequently occuring functions which arose in this way were given names, such as elliptic functions, Bessel functions, and so forth.

Our main goal in this chapter is to study how some of these functions can be constructed via the process of finding solutions to differential equations by means of power series. As a first example, we consider our old friend, the equation of simple harmonic motion

$$\frac{d^2y}{dx^2} + y = 0, \tag{2.11}$$

which we have already learned how to solve by other methods. Suppose for the moment that we don't know the general solution and want to find it by means of power series. We could start by assuming that

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{n=0}^{\infty} a_nx^n. \tag{2.12}$$

It can be shown that the standard technique for differentiating polynomials term by term also works for power series, so we expect that

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \cdots = \sum_{n=1}^{\infty} na_nx^{n-1}.$$

(Note that the last summation only goes from 1 to ∞, since the term with $n = 0$ drops out of the sum.) Differentiating again yields

$$\frac{d^2y}{dx^2} = 2a_2 + 3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + \cdots = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}.$$

We can replace n by $m + 2$ in the last summation so that

$$\frac{d^2y}{dx^2} = \sum_{m+2=2}^{\infty} (m+2)(m+2-1)a_{m+2}x^{m+2-2} = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m.$$

The index m is a "dummy variable" in the summation and can be replaced by any other letter. Thus we are free to replace m by n and obtain the formula

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

Substitution into equation(2.11) yields

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}a_n x^n = 0,$$

or

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + a_n]x^n = 0.$$

Recall that a polynomial is zero only if all its coefficients are zero. Similarly, a power series can be zero only if all of its coefficients are zero. It follows that

$$(n+2)(n+1)a_{n+2} + a_n = 0,$$

or

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}. \tag{2.13}$$

This is called a *recursion formula* for the coefficients a_n.

The first two coefficients a_0 and a_1 in the power series can be determined from the initial conditions,

$$y(0) = a_0, \qquad \frac{dy}{dx}(0) = a_1.$$

Then the recursion formula can be used to determine the remaining coefficients by the process of induction. Indeed it follows from (2.13) with $n = 0$ that

$$a_2 = -\frac{a_0}{2 \cdot 1} = -\frac{1}{2}a_0.$$

Similarly, it follows from (2.13) with $n = 1$ that

$$a_3 = -\frac{a_1}{3 \cdot 2} = -\frac{1}{3!}a_1,$$

and with $n = 2$ that

$$a_4 = -\frac{a_2}{4 \cdot 3} = \frac{1}{4 \cdot 3}\frac{1}{2}a_0 = \frac{1}{4!}a_0.$$

Continuing in this manner, we find that

$$a_{2n} = \frac{(-1)^n}{(2n)!}a_0, \qquad a_{2n+1} = \frac{(-1)^n}{(2n+1)!}a_1.$$

Substitution into (2.12) yields

$$\begin{aligned} y &= a_0 + a_1 x - \frac{1}{2!}a_0 x^2 - \frac{1}{3!}a_1 x^3 + \frac{1}{4!}a_0 x^4 + \cdots \\ &= a_0\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right) + a_1\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right). \end{aligned}$$

We recognize that the expressions within parentheses are power series expansions of the functions $\sin x$ and $\cos x$, and hence we obtain the familiar expression for the solution to the equation of simple harmonic motion,

$$y = a_0 \cos x + a_1 \sin x.$$

The method we have described—assuming a solution to the differential equation of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and solve for the coefficients a_n—is surprisingly effective, particularly for the class of equations called second-order linear differential equations.

It is proven in books on differential equations that if $P(x)$ and $Q(x)$ are well-behaved functions, then the solutions to the "homogeneous linear differential equation"

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

can be organized into a two-parameter family

$$y = a_0 y_0(x) + a_1 y_1(x),$$

called the *general solution*. Here $y_0(x)$ and $y_1(x)$ are any two nonzero solutions, neither of which is a constant multiple of the other. In the terminology used in linear algebra, we say that they are linearly independent solutions. As a_0 and a_1 range over all constants, y ranges throughout a "linear space" of solutions, and we say that $y_0(x)$ and $y_1(x)$ form a *basis* for the space of solutions.

In the special case where the functions $P(x)$ and $Q(x)$ are real analytic, the solutions $y_0(x)$ and $y_1(x)$ will also be real analytic. This is the content of the following theorem, which is proven in more advanced books on differential equations:

Theorem. *If the functions $P(x)$ and $Q(x)$ can be represented by power series*

$$P(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n, \qquad Q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n$$

with positive radii of convergence R_1 and R_2 respectively, then any solution $y(x)$ to the linear differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

can be represented by a power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

whose radius of convergence is $\geq$ the smallest of R_1 and R_2.

This theorem is used to justify the solution of many well-known differential equations by means of the power series method.

Example. *Legendre's differential equation* is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0, \tag{2.14}$$

where p is a parameter. This equation is very useful for treating spherically symmetric potentials in the theories of Newtonian gravitation and in electricity and magnetism.

To apply our theorem, we need to divide by $1-x^2$ to obtain

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{p(p+1)}{1-x^2}y = 0.$$

Thus we have

$$P(x) = -\frac{2x}{1-x^2}, \qquad Q(x) = \frac{p(p+1)}{1-x^2}.$$

Now from the preceding section, we know that the power series

$$1 + u + u^2 + u^3 + \cdots \qquad \text{converges to} \qquad \frac{1}{1-u}$$

for $|u| < 1$. If we substitute $u = x^2$, we can conclude that

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots,$$

the power series converging when $|x| < 1$. It follows quickly that

$$P(x) = -\frac{2x}{1-x^2} = -2x - 2x^3 - 2x^5 - \cdots$$

and

$$Q(x) = \frac{p(p+1)}{1-x^2} = p(p+1) + p(p+1)x^2 + p(p+1)x^4 + \cdots.$$

Both of these functions have power series expansions about $x_0 = 0$ which converge for $|x| < 1$. Hence our theorem implies that any solution to Legendre's equation will be expressible as a power series about $x_0 = 0$ which converges for $|x| < 1$. However, we might suspect that the solutions to Legendre's equation to exhibit some unpleasant behaviour near $x = \pm 1$. Experimentation with numerical solutions to Legendre's equation would show that these suspicions are justified—solutions to Legendre's equation will usually blow up as $x \to \pm 1$.

We can verify these properties of the solutions by solving Legendre's equation by the power series method. As in the case of the equation of simple harmonic motion, we write

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

We differentiate term by term as before, and obtain

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad \frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Once again, we can replace n by $m+2$ in the last summation so that

$$\frac{d^2y}{dx^2} = \sum_{m+2=2}^{\infty} (m+2)(m+2-1) a_{m+2} x^{m+2-2} = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m,$$

and then replace m by n once again, so that

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n. \tag{2.15}$$

Moreover,

$$-x^2 \frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} -n(n-1) a_n x^n, \tag{2.16}$$

$$-2x \frac{dy}{dx} = \sum_{n=0}^{\infty} -2n a_n x^n, \tag{2.17}$$

and

$$p(p+1)y = \sum_{n=0}^{\infty} p(p+1) a_n x^n. \tag{2.18}$$

Adding together (2.15), (2.16), (2.17) and (2.18), we obtain

$$\begin{aligned}(1-x^2)&\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (-n(n-1) - 2n + p(p+1)) a_n x^n.\end{aligned}$$

If y satisfies Legendre's equation, we must have

$$0 = \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + (-n(n+1) + p(p+1)) a_n] x^n.$$

Since the right-hand side is zero for all choices of x, each coefficient must be zero, so

$$(n+2)(n+1) a_{n+2} + (-n(n+1) + p(p+1)) a_n = 0,$$

and we obtain the *recursion formula* for the coefficients of the power series:

$$a_{n+2} = \frac{n(n+1) - p(p+1))}{(n+2)(n+1)} a_n = \frac{-(p-n)(p+n+1)}{(n+2)(n+1)} a_n. \tag{2.19}$$

Just as in the case of the equation of simple harmonic motion, the first two coefficients a_0 and a_1 in the power series can be determined from the initial conditions,

$$y(0) = a_0, \qquad \frac{dy}{dx}(0) = a_1.$$

The recursion formula can be used to determine the remaining coefficients in the power series. Indeed it follows from (2.19) with $n = 0$ that

$$a_2 = \frac{-p(p+1)}{2 \cdot 1} a_0.$$

Similarly, it follows from (2.19) with $n = 1$ that

$$a_3 = \frac{-(p-1)(p+2)}{3 \cdot 2} a_1 = \frac{-(p-1)(p+2)}{3!} a_1,$$

and with $n = 2$ that

$$a_4 = \frac{-(p-2)(p+3)}{4 \cdot 3} a_2 = \frac{(p-2)p(p+1)(p+3)}{4!} a_0.$$

Continuing in this manner, we find that

$$a_5 = \frac{-(p-3)(p+4)}{5 \cdot 4} a_3 = \frac{(p-3)(p-1)(p+2)(p+4)}{5!} a_1,$$

$$a_6 = \frac{-(p-4)(p+5)}{6 \cdot 5} a_4 = \frac{-(p-4)(p-2)p(p+1)(p+3)(p+5}{6!} a_0,$$

and so forth. Thus we find that

$$y = a_0 \left[1 - \frac{p(p+1)}{2 \cdot 1} x^2 + \frac{(p-2)p(p+1)(p+3)}{4!} x^4 - \cdots \right] \\ + a_1 \left[x - \frac{2(p-1)}{3!} x^3 + \frac{2^2(p-1)(p-3)}{5!} x^5 - \cdots \right].$$

We can now write the general solution to Legendre's equation in the form

$$y = a_0 y_0(x) + a_1 y_1(x),$$

where

$$y_0(x) = 1 - \frac{p(p+1)}{2 \cdot 1} x^2 + \frac{(p-2)p(p+1)(p+3)}{4!} x^4 - \cdots$$

and

$$y_1(x) = x - \frac{2(p-1)}{3!} x^3 + \frac{2^2(p-1)(p-3)}{5!} x^5 - \cdots .$$

For a given choice of the parameter p, we could use these power series to construct tables of values for the functions $y_0(x)$ and $y_1(x)$. Tables of values for these functions are found in many "handbooks of mathematical functions." In the language of linear algebra, we say that $y_0(x)$ and $y_1(x)$ form a basis for the space of solutions to Legndre's differential equation.

When p is a positive integer, the recursion formula (2.19) implies that $a_{p+2} = 0$. Thus one of the two power series solutions will collapse, yielding a polynomial solution to Legendre's equation. (A polynomial is just a power series that has only finitely many nonzero terms.) Properly normalized, these polynomial solutions are known as *Legendre polynomials*. Of course, polynomial solutions have infinite radius of convergence and are therefore well-behaved for all values of x.

For example, the polynomial

$$y_0(x) = 1 - 3x^2 \qquad \text{or} \qquad P_2(x) = -\frac{1}{2}y_0(x) = \frac{1}{2}(3x^2 - 1)$$

is a solution to Legendre's differential equation with $p = 2$, and $P_2(x)$ is the Legendre polynomial of degree two.

On the other hand, setting $x = 1$ in the solution $y_0(x)$ yields

$$y_0(1) = a_0 + a_2 + a_4 + \cdots = 1 - \frac{2(p-1)}{3!} + \frac{2^2(p-1)(p-3)}{5!} - \cdots . \qquad (2.20)$$

We can ask whether this series solution converges when the power series solution does not reduce to a polynomial. Note that it follows from the recursion formula (2.19) that when n is larger than p,

$$\frac{a_{n+2}}{a_n} = \frac{(n-p)(n+p+1)}{(n+2)(n+1)}$$

is positive, and indeed it is easily checked that

$$\lim_{n\to\infty} \frac{a_{n+2}}{a_n} = 1.$$

With sufficient effort one can use this to show that the series cannot converge at $x = 1$. By a similar argument the infinite series for $y_1(1)$ diverges as well. Note that

$$y_0(-x) = y_0(x), \quad y_1(-x) = -y_1(x).$$

Using these facts, one can show that if neither $y_0(x)$ or $y_1(x)$ reduces to a polynomial, then no linear combination $c_0y_0(x) + c_1y_1(x)$ has a power series which converges at both $x = 1$ and $x = -1$.

This illustrates the power of the power series method. An analysis of the power series solutions to Legendre's differential equation shows that the only solutions to this equation which are well-behaved for all x such that $-1 \le x \le 1$ are the Legendre polynomials, a fact which turns out to be tremendously important for applications.

Exercises:

2.4.1. We would like to use the power series method to find the general solution to the differential equation

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 12y = 0,$$

so we assume the solution is of the form

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

a power series centered at 0, and determine the coefficients a_n.

a. As a first step, find the recursion formula for a_{n+2} in terms of a_n.

b. The coefficients a_0 and a_1 will be determined by the initial conditions. Use the recursion formula to determine a_n in terms of a_0 and a_1, for $2 \leq n \leq 9$.

c. Find a nonzero polynomial solution to this differential equation.

d. Find a basis for the space of solutions to the equation.

e. Find the solution to the initial value problem

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 12y = 0, \qquad y(0) = 0, \qquad \frac{dy}{dx}(0) = 1.$$

f. To solve the differential equation

$$\frac{d^2y}{dx^2} - 4(x-3)\frac{dy}{dx} + 12y = 0,$$

it would be most natural to assume that the solution has the form

$$y = \sum_{n=0}^{\infty} a_n (x-3)^n.$$

Use this idea to find a polynomial solution to the differential equation

$$\frac{d^2y}{dx^2} - 4(x-3)\frac{dy}{dx} + 12y = 0.$$

2.4.2. We want to use the power series method to find the general solution to *Hermite's differential equation*

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2py = 0,$$

where p is a parameter. Once again our approach is to assume our solution is a power series centered at 0 and determine the coefficients in this power series.

a. As a first step, find the recursion formula for a_{n+2} in terms of a_n.

b. Use the recursion formula to determine a_n in terms of a_0 and a_1, for $2 \leq n \leq 9$.

c. Find a nonzero polynomial solution to this differential equation, in the case where $p = 3$.

d. Find a basis for the space of solutions to the differential equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 6y = 0.$$

e. Find the solution to the initial value problem

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 6y = 0, \qquad y(0) = 0, \qquad \frac{dy}{dx}(0) = 1.$$

2.4.3. The differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0,$$

where p is a constant, is known as *Chebyshev's equation*. It can be rewritten in the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \qquad \text{where} \qquad P(x) = -\frac{x}{1-x^2}, \quad Q(x) = \frac{p^2}{1-x^2}.$$

a. If $P(x)$ and $Q(x)$ are represented as power series about $x_0 = 0$, what is the radius of convergence of these power series?

b. Assuming a power series solution centered at 0, find the recursion formula for a_{n+2} in terms of a_n.

c. Use the recursion formula to determine a_n in terms of a_0 and a_1, for $2 \leq n \leq 9$.

d. In the special case where $p = 3$, find a nonzero polynomial solution to this differential equation.

e. Find a basis for the space of solutions to

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 9y = 0.$$

2.4.4. The differential equation

$$\left(-\frac{d^2}{dx^2} + x^2\right) z = \lambda z \tag{2.21}$$

arises when treating the quantum mechanics of simple harmonic motion.

a. Show that making the substitution $z = e^{-x^2/2}y$ transforms this equation into Hermite's differential equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (\lambda - 1)y = 0.$$

b. Show that if $\lambda = 2n+1$ where n is a nonnegative integer, (2.21) has a solution of the form $z = e^{-x^2/2}P_n(x)$, where $P_n(x)$ is a polynomial.

2.4.5. The differential equation with dependent variable w,

$$(1-x^2)^2\frac{d^2w}{dx^2} - 2x(1-x^2)\frac{dw}{dx} + (-\lambda(1-x^2) - m^2)w = 0,$$

is known as an *associated Legendre differential equation*, whenever m is an integer.

a. Show that if $y(x)$ is a solution to Legendre's equation, then

$$w(x) = \sqrt{1-x^2}\,y'(x)$$

is a solution to the associated Legendre equation with $m = 1$.

b. Show that if $y(x)$ is a solution to Legendre's equation, then

$$w(x) = (1-x^2)^{m/2}\frac{d^my}{dx^m}(x)$$

is a solution to the associated Legendre equation for general values of m.

2.5 Singular points

Our ultimate goal is to give a mathematical description of the vibrations of a circular drum. For this, we will need to solve Bessel's equation, a second-order homogeneous linear differential equation with a "singular point" at 0.

A point x_0 is called an *ordinary point* for the differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \tag{2.22}$$

if the coefficients $P(x)$ or $Q(x)$ are both real analytic at $x = x_0$, or equivalently, both $P(x)$ or $Q(x)$ have power series expansions about $x = x_0$ with positive radius of convergence. In the opposite case, we say that x_0 is a *singular point*; thus x_0 is a singular point if at least one of the coefficients $P(x)$ or $Q(x)$ fails to be real analytic at $x = x_0$. A singular point is said to be *regular* if

$$(x - x_0)P(x) \qquad \text{and} \qquad (x-x_0)^2Q(x)$$

are real analytic. Otherwise, the singular point is said to be *irregular*.

For example, $x_0 = 1$ is a singular point for Legendre's equation

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{p(p+1)}{1-x^2}y = 0,$$

because $1 - x^2 \to 0$ as $x \to 1$ and hence the quotients

$$\frac{2x}{1-x^2} \qquad \text{and} \qquad \frac{p(p+1)}{1-x^2}$$

blow up as $x \to 1$, but it is a regular singular point because

$$(x-1)P(x) = (x-1)\frac{-2x}{1-x^2} = \frac{2x}{x+1}$$

and

$$(x-1)^2 Q(x) = (x-1)^2 \frac{p(p+1)}{1-x^2} = \frac{p(p+1)(1-x)}{1+x}$$

are both real analytic at $x_0 = 1$.

The point of these definitions is that in the case where $x = x_0$ is a regular singular point, a modification of the power series method can still be used to find solutions.

Theorem 1. *If x_0 is a regular singular point for the differential equation*

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0,$$

then this differential equation has at least one nonzero solution of the form

$$y(x) = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n, \tag{2.23}$$

where $a_0 \neq 0$ and r is a constant, which may be complex. If $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ have power series which converge for $|x-x_0| < R$, then the power series

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

will also converge for $|x-x_0| < R$.

We will call a solution of the form (2.23) a *generalized power series solution*. Unfortunately, the theorem guarantees only one generalized power series solution, not a basis. In fortuitous cases, one can find a basis of generalized power series solutions, but not always. The method of finding generalized power series solutions to (2.22) in the case of regular singular points is called the *Frobenius method*.[2]

The simplest differential equation to which the Theorem of Frobenius applies is the Cauchy-Euler equidimensional equation. This is the special case of (2.22) for which

$$P(x) = \frac{p}{x}, \qquad Q(x) = \frac{q}{x^2},$$

where p and q are constants. Note that

$$xP(x) = p \qquad \text{and} \qquad x^2Q(x) = q$$

[2]For more discussion of the Frobenius method as well as many of the other techniques touched upon in this chapter we refer the reader to George F. Simmons, *Differential equations with applications and historical notes*, second edition, McGraw-Hill, New York, 1991.

are real analytic, so $x = 0$ is a regular singular point for the Cauchy-Euler equation as long as either p or q is nonzero.

The Frobenius method is quite simple in the case of Cauchy-Euler equations. Indeed, in this case, we can simply take $y(x) = x^r$, substitute into the equation and solve for r. Often there will be two linearly independent solutions $y_1(x) = x^{r_1}$ and $y_2(x) = x^{r_2}$ of this special form. In this case, the general solution is given by the superposition principle as

$$y = c_1 x^{r_1} + c_2 x^{r_2}.$$

For example, to solve the differential equation

$$x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0,$$

we set $y = x^r$ and differentiate to show that

$$\begin{array}{ll} dy/dx = rx^{r-1} & \Rightarrow x(dy/dx) = rx^r, \\ d^2y/dx^2 = r(r-1)x^{r-2} & \Rightarrow x^2(d^2y/dx^2) = r(r-1)x^r. \end{array}$$

Substitution into the differential equation yields

$$r(r-1)x^r + 4rx^r + 2x^r = 0,$$

and dividing by x^r yields

$$r(r-1) + 4r + 2 = 0 \qquad \text{or} \qquad r^2 + 3r + 2 = 0.$$

The roots to this equation are $r = -1$ and $r = -2$, so the general solution to the differential equation is

$$y = c_1 x^{-1} + c_2 x^{-2} = \frac{c_1}{x} + \frac{c_2}{x^2}.$$

Note that the solutions $y_1(x) = x^{-1}$ and $y_2(x) = x^{-2}$ can be rewritten in the form

$$y_1(x) = x^{-1}\sum_{n=0}^{\infty} a_n x^n, \qquad y_2(x) = x^{-2}\sum_{n=0}^{\infty} b_n x^n,$$

where $a_0 = b_0 = 1$ and all the other a_n's and b_n's are zero, so both of these solutions are generalized power series solutions.

On the other hand, if this method is applied to the differential equation

$$x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = 0,$$

we obtain

$$r(r-1) + 3r + 1 = r^2 + 2r + 1,$$

which has a repeated root. In this case, we obtain only a one-parameter family of solutions

$$y = cx^{-1}.$$

Fortunately, there is a trick that enables us to handle this situation, the so-called method of variation of parameters. In this context, we replace the parameter c by a variable $v(x)$ and write

$$y = v(x)x^{-1}.$$

Then

$$\frac{dy}{dx} = v'(x)x^{-1} - v(x)x^{-2}, \qquad \frac{d^2y}{dx^2} = v''(x)x^{-1} - 2v'(x)x^{-2} + 2v(x)x^{-3}.$$

Substitution into the differential equation yields

$$x^2(v''(x)x^{-1} - 2v'(x)x^{-2} + 2v(x)x^{-3}) + 3x(v'(x)x^{-1} - v(x)x^{-2}) + v(x)x^{-1} = 0,$$

which quickly simplifies to yield

$$xv''(x) + v'(x) = 0, \qquad \frac{v''}{v'} = -\frac{1}{x}, \qquad \log|v'| = -\log|x| + a, \qquad v' = \frac{c_2}{x},$$

where a and c_2 are constants of integration. A further integration yields

$$v = c_2 \log|x| + c_1, \qquad \text{so} \qquad y = (c_2 \log|x| + c_1)x^{-1},$$

and we obtain the general solution

$$y = c_1\frac{1}{x} + c_2\frac{\log|x|}{x}.$$

In this case, only one of the basis elements in the general solution is a generalized power series.

For equations which are not of Cauchy-Euler form the Frobenius method is more involved. Let us consider the example

$$2x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0, \tag{2.24}$$

which can be rewritten as

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \qquad \text{where} \qquad P(x) = \frac{1}{2x}, \qquad Q(x) = \frac{1}{2x}.$$

From this one sees that $x = 0$ is a regular singular point. We begin the Frobenius method by assuming that the solution has the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Then

$$\frac{dy}{dx} = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}, \qquad \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

and

$$2x\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r-1}.$$

Substitution into the differential equation yields

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0,$$

which simplifies to

$$x^r \left[\sum_{n=0}^{\infty}(2n+2r-1)(n+r)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n\right] = 0.$$

We can divide by x^r, and separate out the first term from the first summation, obtaining

$$(2r-1)ra_0 x^{-1} + \sum_{n=1}^{\infty}(2n+2r-1)(n+r)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

If we let $n = m+1$ in the first infinite sum, this becomes

$$(2r-1)ra_0 x^{-1} + \sum_{m=0}^{\infty}(2m+2r+1)(m+r+1)a_{m+1} x^m + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Finally, we replace m by n, obtaining

$$(2r-1)ra_0 x^{-1} + \sum_{n=0}^{\infty}(2n+2r+1)(n+r+1)a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

The coefficient of each power of x must be zero. In particular, we must have

$$(2r-1)ra_0 = 0, \qquad (2n+2r+1)(n+r+1)a_{n+1} + a_n = 0. \tag{2.25}$$

If $a_0 = 0$, then all the coefficients must be zero from the second of these equations,and we don't get a nonzero solution. So we must have $a_0 \neq 0$ and hence

$$(2r-1)r = 0.$$

This is called the *indicial equation*. In this case, it has two roots

$$r_1 = 0, \qquad r_2 = \frac{1}{2}.$$

The second half of (2.25) yields the recursion formula

$$a_{n+1} = -\frac{1}{(2n+2r+1)(n+r+1)}a_n, \qquad \text{for} \qquad n \geq 0.$$

We can try to find a generalized power series solution for either root of the indicial equation. If $r = 0$, the recursion formula becomes

$$a_{n+1} = -\frac{1}{(2n+1)(n+1)} a_n.$$

Given $a_0 = 1$, we find that

$$a_1 = -1, \qquad a_2 = -\frac{1}{3 \cdot 2} a_1 = \frac{1}{3 \cdot 2},$$

$$a_3 = -\frac{1}{5 \cdot 3} a_2 = -\frac{1}{(5 \cdot 3)(3 \cdot 2)}, \qquad a_4 = -\frac{1}{7 \cdot 4} a_3 = \frac{1}{(7 \cdot 5 \cdot 3)4!},$$

and so forth. In general, we would have

$$a_n = (-1)^n \frac{1}{(2n-1)(2n-3) \cdots 1 \cdot n!}.$$

One of the generalized power series solution to (2.24) is

$$y_1(x) = x^0 \left[1 - x + \frac{1}{3 \cdot 2} x^2 - \frac{1}{(5 \cdot 3)(3!)} x^3 + \frac{1}{(7 \cdot 5 \cdot 3)4!} x^4 - \cdots \right]$$

$$= 1 - x + \frac{1}{3 \cdot 2} x^2 - \frac{1}{(5 \cdot 3)(3!)} x^3 + \frac{1}{(7 \cdot 5 \cdot 3)4!} x^4 - \cdots .$$

If $r = 1/2$, the recursion formula becomes

$$a_{n+1} = -\frac{1}{(2n+2)(n+(1/2)+1)} a_n = -\frac{1}{(n+1)(2n+3)} a_n.$$

Given $a_0 = 1$, we find that

$$a_1 = -\frac{1}{3}, \qquad a_2 = -\frac{1}{2 \cdot 5} a_1 = \frac{1}{2 \cdot 5 \cdot 3},$$

$$a_3 = -\frac{1}{3 \cdot 7} a_2 = -\frac{1}{3! \cdot (7 \cdot 5 \cdot 3)},$$

and in general,

$$a_n = (-1)^n \frac{1}{n! \cdot (2n+1)(2n-1) \cdots 1}.$$

We thus obtain a second generalized power series solution to (2.24):

$$y_2(x) = x^{1/2} \left[1 - \frac{1}{3} x + \frac{1}{2 \cdot 5 \cdot 3} x^2 - \frac{1}{3! \cdot (7 \cdot 5 \cdot 3)} x^3 + \cdots \right].$$

The general solution to (2.24) is a superposition of $y_1(x)$ and $y_2(x)$:

$$y = c_1 \left[1 - x + \frac{1}{3 \cdot 2} x^2 - \frac{1}{(5 \cdot 3)(3!)} x^3 + \frac{1}{(7 \cdot 5 \cdot 3)4!} x^4 - \cdots \right]$$

$$+c_2\sqrt{x}\left[1-\frac{1}{3}x+\frac{1}{2\cdot 5\cdot 3}x^2-\frac{1}{3!\cdot(7\cdot 5\cdot 3)}x^3+\cdots\right].$$

We obtained two linearly independent generalized power series solutions in this case. Indeed, we can always find two linearly independent solutions when the roots r_1 and r_2 of the indicial equation do not differ by an integer:

Theorem 2. *If x_0 is a regular singular point for the differential equation*

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0,$$

and the roots r_1 and r_2 of the indicial equation do not differ from an integer, there are two linearly independent solutions of the form (2.23), where $a_0 \neq 0$ and $r = r_1$ or r_2.

If the roots of the indicial equation differ by an integer, we may obtain only one generalized power series solution. In that case, the rule of thumb is that the r occurring in (2.23) should be chosen to be the larger of the two roots. A second independent solution can then be found by variation of parameters, just as we saw in the case of the Cauchy-Euler equidimensional equation.

For example, let us consider the differential equation

$$x\frac{d^2y}{dx^2}-2\alpha x\frac{dy}{dx}+2y=0,$$

which arises in quantum mechanics when studying the wave function for the hydrogen atom.[3] This equation can be rewritten as

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0, \qquad \text{where} \qquad P(x)=-2\alpha, \qquad Q(x)=\frac{2}{x}.$$

Once again, $x = 0$ is a regular singular point. We begin the Frobenius method by assuming that the solution has the form

$$y=x^r\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_nx^{n+r}.$$

Then

$$\frac{dy}{dx}=\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}, \qquad \frac{d^2y}{dx^2}=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}$$

and

$$x\frac{d^2y}{dx^2}=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-1}, \qquad x\frac{dy}{dx}=\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}.$$

[3]See §19.2 in Feynman, Leighton and Sands, *The Feynman lectures on physics*, Volume III, Addison-Wesley, New York, 1965.

Substitution into the differential equation yields

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=0}^{\infty} 2\alpha(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0,$$

which simplifies to

$$r(r-1)a_0 x^{r-1} + x^r \left[\sum_{n=0}^{\infty}(n+r)(n+r+1)a_{n+1}x^{n+r} \right.$$
$$\left. - \sum_{n=0}^{\infty}(2\alpha(n+r)-2)a_n x^{n+r} \right] = 0.$$

Thus in this case, the condition that $a_0 \neq 0$ implies that the indicial equation $r(r-1) = 0$ must hold, and it has roots $r = 0$ and $r = 1$. Our rule of thumb is to choose $r = 1$, and then the above expression becomes

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty}(2\alpha(n+1)-2)a_n x^{n+1} = 0.$$

This yields the recursion formula

$$a_{n+1} = \frac{2\alpha(n+1)-2}{(n+2)(n+1)}a_n = \frac{2((n+1)\alpha - 1)}{(n+2)(n+1)}a_n.$$

Given an arbitrary a_0, we can determine the other coefficients a_n from this recursion formula, obtaining by a short calculation

$$a_n = 2^n \frac{(n\alpha - 1)\cdots(2\alpha-1)(\alpha-1)}{(n+1)(n!)^2}a_0.$$

This yields a nontrivial solution

$$y = x\sum_{n=0}^{\infty} a_n x^n$$

to the differential equation. The second independent solution could then be found by variation of parameters, as we did in the case of the Cauchy-Euler equation when the characteristic equation has repeated roots.

Exercises:

2.5.1. For each of the following differential equations, determine whether $x = 0$ is ordinary or singular. If it is singular, determine whether it is regular or not.

a. $y'' + xy' + (1-x^2)y = 0$.

b. $y'' + (1/x)y' + (1-(1/x^2))y = 0$.

c. $x^2y'' + 2xy' + (\cos x)y = 0$.

d. $x^3y'' + 2xy' + (\cos x)y = 0$.

2.5.2. Find the general solution to each of the following Cauchy-Euler equations:

a. $x^2d^2y/dx^2 - 2xdy/dx + 2y = 0$.

b. $x^2d^2y/dx^2 - xdy/dx + y = 0$.

c. $x^2d^2y/dx^2 - xdy/dx + 10y = 0$.

(Hint: Use the formula

$$x^{a+bi} = x^a x^{bi} = x^a(e^{\log x})^{bi} = x^a e^{ib\log x} = x^a[\cos(b\log x) + i\sin(b\log x)]$$

to simplify the answer.)

d. $(x+1)^2d^2y/dx^2 - 2(x+1)xdy/dx + 2y = 0$.

(Hint: Try solutions of the form $y(x) = (x+1)^r$.)

2.5.3. We want to find generalized power series solutions to the differential equation

$$3x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

by the method of Frobenius. Our procedure is to find solutions of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r},$$

where r and the a_n's are constants.

a. Determine the indicial equation and the recursion formula.

b. Find two linearly independent generalized power series solutions.

2.5.4. To find generalized power series solutions to the differential equation

$$2x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$$

by the method of Frobenius, we assume the solution has the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r},$$

where r and the a_n's are constants.

a. Determine the indicial equation and the recursion formula.

b. Find two linearly independent generalized power series solutions.

2.5.5. Determine whether $x = \pm 1$ are regular singular points for the associated Legendre equation

$$(1-x^2)^2 \frac{d^2y}{dx^2} - 2x(1-x^2)\frac{dy}{dx} + (-\lambda(1-x^2) - m^2)y = 0,$$

2.6 Bessel's differential equation*

Our next goal is to apply the Frobenius method to *Bessel's equation*,

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (x^2 - p^2)y = 0, \tag{2.26}$$

an equation which is needed to analyze the vibrations of a circular drum, as we mentioned before. Here p is a parameter, which will be a nonnegative integer in the vibrating drum problem. Using the Leibniz rule for differentiating a product, we can rewrite Bessel's equation in the form

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0$$

or equivalently as

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x) = 0,$$

where

$$P(x) = \frac{1}{x} \qquad \text{and} \qquad Q(x) = \frac{x^2 - p^2}{x^2}.$$

Since

$$xP(x) = 1 \qquad \text{and} \qquad x^2 Q(x) = x^2 - p^2,$$

we see that $x = 0$ is a regular singular point, so the Frobenius theorem implies that there exists at least one nonzero generalized power series solution to (2.26).

In the applications of Bessel's equation, we often want a solution that is well-behaved near $x = 0$. To find such a solution, we start as in the previous section by assuming that

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Then

$$\frac{dy}{dx} = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}, \qquad x\frac{dy}{dx} = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r},$$

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) = \sum_{n=0}^{\infty}(n+r)^2 a_n x^{n+r-1},$$

and thus

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) = \sum_{n=0}^{\infty}(n+r)^2 a_n x^{n+r}. \tag{2.27}$$

On the other hand,

$$x^2 y = \sum_{n=0}^{\infty} a_n x^{n+r+2} = \sum_{m=2}^{\infty} a_{m-2} x^{m+r},$$

where we have set $m = n + 2$. Replacing m by n then yields

$$x^2 y = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}. \tag{2.28}$$

Finally, we have,

$$-p^2 y = -\sum_{n=0}^{\infty} p^2 a_n x^{n+r}. \tag{2.29}$$

Adding up (2.27), (2.28), and (2.29), we find that if y is a solution to (2.26),

$$\sum_{n=0}^{\infty} (n+r)^2 a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} - \sum_{n=0}^{\infty} p^2 a_n x^{n+r} = 0.$$

This simplifies to yield

$$\sum_{n=0}^{\infty} [(n+r)^2 - p^2] a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0,$$

or after division by x^r,

$$\sum_{n=0}^{\infty} [(n+r)^2 - p^2] a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

Thus we find that

$$(r^2 - p^2) a_0 + [(r+1)^2 - p^2] a_1 x + \sum_{n=2}^{\infty} \{[(n+r)^2 - p^2] a_n + a_{n-2}\} x^n = 0.$$

The coefficient of each power of x must be zero, so

$$(r^2 - p^2) a_0 = 0, \quad [(r+1)^2 - p^2] a_1 = 0, \quad [(n+r)^2 - p^2] a_n + a_{n-2} = 0 \quad \text{for } n \geq 2.$$

Since we want a_0 to be nonzero, r must satisfy the *indicial equation*

$$(r^2 - p^2) = 0,$$

which implies that $r = \pm p$. We want a solution which is as well-behaved near $x = 0$ as possible, so we take $p \geq 0$ and $r = p$. (Recall that the second independent solution can then be obtained by variation of parameters if necessary.) Then

$$[(p+1)^2 - p^2] a_1 = 0 \qquad \Rightarrow \qquad (2p+1) a_1 = 0 \qquad \Rightarrow \qquad a_1 = 0.$$

Finally,

$$[(n+p)^2 - p^2]a_n + a_{n-2} = 0 \qquad \Rightarrow \qquad [n^2 + 2np]a_n + a_{n-2} = 0,$$

which yields the recursion formula

$$a_n = -\frac{1}{2np + n^2} a_{n-2}. \tag{2.30}$$

The recursion formula implies that $a_n = 0$ if n is odd.

In the special case where p is a nonnegative integer, we will get a genuine power series solution to Bessel's equation (2.26). Let us focus now on this important case. We could of course set a_0 equal to any nonzero constant that we want, but the final formula we obtain will look much nicer if we set

$$a_0 = \frac{1}{2^p p!}.$$

It then follows from the recursion formula that

$$a_2 = \frac{-a_0}{4p+4} = -\frac{1}{4(p+1)} \frac{1}{2^p p!} = (-1)\left(\frac{1}{2}\right)^{p+2} \frac{1}{1!(p+1)!},$$

$$\begin{aligned} a_4 = \frac{-a_2}{8p+16} &= \frac{1}{8(p+2)} \left(\frac{1}{2}\right)^{p+2} \frac{1}{1!(p+1)!} \\ &= \frac{1}{2(p+2)} \left(\frac{1}{2}\right)^{p+4} \frac{1}{1!(p+1)!} = (-1)^2 \left(\frac{1}{2}\right)^{p+4} \frac{1}{2!(p+2)!}, \end{aligned}$$

and so forth. The general term is

$$a_{2m} = (-1)^m \left(\frac{1}{2}\right)^{p+2m} \frac{1}{m!(p+m)!}.$$

Thus we finally obtain the power series solution

$$y = \left(\frac{x}{2}\right)^p \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!(p+m)!} \left(\frac{x}{2}\right)^{2m}.$$

The function defined by the power series on the right-hand side is called the p-th *Bessel function of the first kind*, and is denoted by the symbol $J_p(x)$. For example,

$$J_0(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{(m!)^2} \left(\frac{x}{2}\right)^{2m}.$$

Using the comparison and ratio tests, we can show that the power series expansion for $J_p(x)$ has infinite radius of convergence. Thus when p is an integer, Bessel's equation has a nonzero solution which is real analytic at $x = 0$.

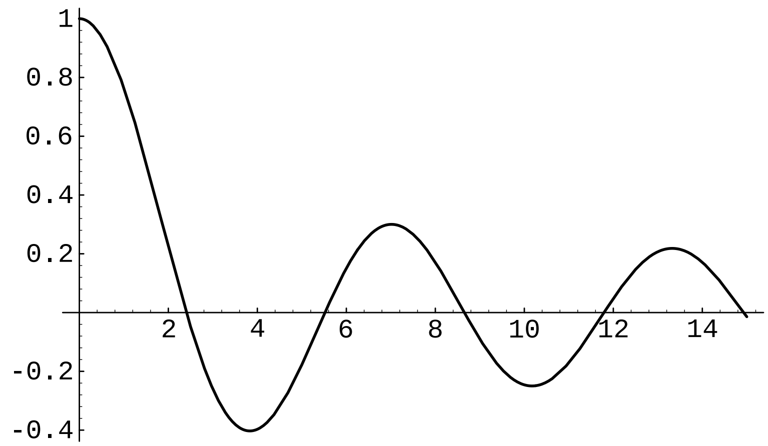

Figure 2.1: Graph of the Bessel function $J_0(x)$.

Bessel functions are so important that Mathematica includes them in its library of built-in functions.[4] Mathematica represents the Bessel functions of the first kind symbolically by `BesselJ[n,x]`. Thus to plot the Bessel function $J_n(x)$ on the interval $[0, 15]$ one simply types in

```
n=0; Plot[ BesselJ[n,x], {x,0,15}]
```

and a plot similar to that of Figure 1.1 will be produced. Similarly, we can plot $J_n(x)$, for $n = 1, 2, 3 \ldots$. Note that the graph of $J_0(x)$ suggests that it has infinitely many positive zeros.

On the open interval $0 < x < \infty$, Bessel's equation has a two-dimensional space of solutions. However, it turns out that when p is a nonnegative integer, a second solution, linearly independent from the Bessel function of the first kind, cannot be obtained directly by the generalized power series method that we have presented. To obtain a basis for the space of solutions, we can, however, apply the method of variation of parameters just as we did in the previous section for the Cauchy-Euler equation when the characteristic equation has repeated roots; namely, we can set

$$y = v(x)J_p(x),$$

substitute into Bessel's equation and solve for $v(x)$. If we were to carry this out in detail, we would obtain a two-parameter family of solutions generated by $J_p(x)$ and another function $Y_p(x)$, called the p-th *Bessel function of the second kind*. Unlike the Bessel function of the first kind, the Bessel function of the second kind is not well-behaved near $x = 0$.

To see why, suppose that $y_1(x)$ and $y_2(x)$ is a basis for the solutions on the

[4]For a very brief introduction to Mathematica, the reader can refer to Appendix A.

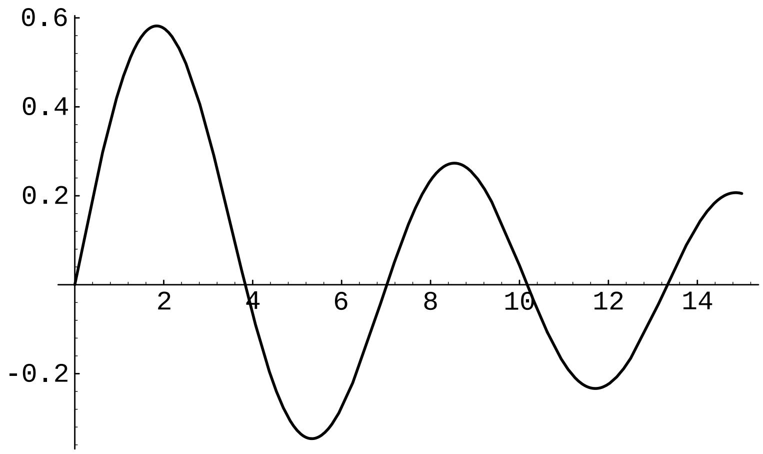

Figure 2.2: Graph of the Bessel function $J_1(x)$.

interval $0 < x < \infty$, and let $W(y_1, y_2)$ be their *Wronskian*, defined by

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_1'(x) \\ y_2(x) & y_2'(x) \end{vmatrix}.$$

This Wronskian must satisfy the first order equation

$$\frac{d}{dx}(xW(y_1, y_2)(x)) = 0,$$

as one verifies by a direct calculation:

$$\begin{aligned} x\frac{d}{dx}(xy_1y_2' - xy_2y_1') &= y_1x\frac{d}{dx}(xy_2') - y_2x\frac{d}{dx}(xy_1') \\ &= -(x^2 - n^2)(y_1y_2 - y_2y_1) = 0. \end{aligned}$$

Thus

$$xW(y_1, y_2)(x) = c, \qquad \text{or} \qquad W(y_1, y_2)(x) = \frac{c}{x},$$

where c is a nonzero constant, an expression which is unbounded as $x \to 0$. It follows that two linearly independent solutions $y_1(x)$ and $y_2(x)$ to Bessel's equation cannot both be well-behaved at $x = 0$.

Let us summarize what we know about the space of solutions to Bessel's equation in the case where p is an integer:

- There is a one-dimensional space of real analytic solutions to (2.26), which are well-behaved as $x \to 0$.
- This one-dimensional space is generated by the Bessel function $J_p(x)$ which is given by the explicit power series formula

$$J_p(x) = \left(\frac{x}{2}\right)^p \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!(p+m)!} \left(\frac{x}{2}\right)^{2m}.$$

Exercises:

2.6.1. Using the explicit power series formulae for $J_0(x)$ and $J_1(x)$ show that

$$\frac{d}{dx}J_0(x) = -J_1(x) \qquad \text{and} \qquad \frac{d}{dx}(xJ_1(x)) = xJ_0(x).$$

2.6.2. The differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + p^2)y = 0$$

is sometimes called a *modified Bessel equation*. Find a generalized power series solution to this equation in the case where p is an integer. (Hint: The power series you obtain should be very similar to the power series for $J_p(x)$.)

2.6.3. Show that the functions

$$y_1(x) = \frac{1}{\sqrt{x}}\cos x \qquad \text{and} \qquad y_2(x) = \frac{1}{\sqrt{x}}\sin x$$

are solutions to Bessel's equation

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (x^2 - p^2)y = 0,$$

in the case where $p = 1/2$. Hence the general solution to Bessel's equation in this case is

$$y = c_1\frac{1}{\sqrt{x}}\cos x + c_2\frac{1}{\sqrt{x}}\sin x.$$

2.6.4. To obtain a nice expression for the generalized power series solution to Bessel's equation in the case where p is not an integer, it is convenient to use the *gamma function* defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

a. Use integration by parts to show that $\Gamma(x+1) = x\Gamma(x)$.

b. Show that $\Gamma(1) = 1$.

c. Show that

$$\Gamma(n+1) = n! = n(n-1)\cdots 2\cdot 1,$$

when n is a positive integer.

d. Set

$$a_0 = \frac{1}{2^p\Gamma(p+1)},$$

and use the recursion formula (2.30) to obtain the following generalized power series solution to Bessel's equation (2.26) for general choice of p:

$$y = J_p(x) = \left(\frac{x}{2}\right)^p \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!\Gamma(p+m+1)} \left(\frac{x}{2}\right)^{2m}.$$

Chapter 3

Fourier Series

3.1 Fourier series

The theory of Fourier series and the Fourier transform is concerned with dividing a function into a superposition of sines and cosines, its components of various frequencies. It is a crucial tool for understanding waves, including water waves, sound waves and light waves. Suppose, for example, that the function $f(t)$ represents the amplitude of a light wave arriving from a distant galaxy. The light is a superposition of many frequencies which encode information regarding the material which makes up the stars of the galaxy, the speed with which the galaxy is receding from the earth, its speed of rotation, and so forth. Much of our knowledge of the universe is derived from analyzing the spectra of stars and galaxies. Just as a prism or a spectrograph is an experimental apparatus for dividing light into its components of various frequencies, so Fourier analysis is a mathematical technique which enables us to decompose an arbitrary function into a superposition of oscillations.

In this chapter, we will describe how the theory of Fourier series can be used to analyze the flow of heat in a bar and the motion of a vibrating string. Fourier series are named for Joseph Fourier, who encountered them in his attempts to understand heat flow.[1] Nowadays, the notion of dividing a function into its components with respect to an appropriate "orthonormal basis of functions" is one of the key ideas of applied mathematics, useful not only as a tool for solving partial differential equations, as we will see in the next two chapters, but for many other purposes as well.

We begin with the basics of Fourier analysis in its simplest context. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *periodic* of period T if it satisfies the condition

$$f(t+T) = f(t), \quad \text{for all } t \in \mathbb{R}.$$

For example, the function $f(t) = \sin t$ is periodic of period 2π.

[1] Fourier's research was published in his *Théorie analytique de la chaleur* in 1822.

Given an arbitrary period T, it is easy to construct examples of functions which are periodic of period T from the trigonometric functions—indeed, the function $f(t) = \sin(\frac{2\pi t}{T})$ is periodic of period T because

$$\sin(\frac{2\pi(t+T)}{T}) = \sin(\frac{2\pi t}{T} + 2\pi) = \sin(\frac{2\pi t}{T}).$$

More generally, if k is any positive integer, the functions

$$\cos(\frac{2\pi kt}{T}) \quad \text{and} \quad \sin(\frac{2\pi kt}{T})$$

are also periodic functions of period T.

The main theorem from the theory of Fourier series states that any "well-behaved" periodic function of period T can be expressed as a superposition of sines and cosines:

$$f(t) = \frac{a_0}{2} + a_1\cos(\frac{2\pi t}{T}) + a_2\cos(\frac{4\pi t}{T}) + \ldots + b_1\sin(\frac{2\pi t}{T}) + b_2\sin(\frac{4\pi t}{T}) + \ldots. \quad (3.1)$$

In this formula, the a_k's and b_k's are called the *Fourier coefficients* of f, and the infinite series on the right-hand side is called the *Fourier series* of f.

Our first goal is to determine how to calculate these Fourier coefficients. For simplicity, we will restrict our attention to the case where the period $T = 2\pi$, so that

$$f(t) = \frac{a_0}{2} + a_1\cos t + a_2\cos 2t + \ldots + b_1\sin t + b_2\sin 2t + \ldots. \quad (3.2)$$

The formulae for a general period T are only a little more complicated, and are based upon exactly the same ideas.

Expression (3.2) is an infinite series, just like the power series studied in Chapter 1, and it is sometimes easier to write it as

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kt + \sum_{k=1}^{\infty} b_k \sin kt.$$

The coefficient a_0 is particularly easy to evaluate. We simply integrate both sides of (3.2) from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(t)dt = \int_{-\pi}^{\pi} \frac{a_0}{2}dt + \int_{-\pi}^{\pi} a_1\cos t dt + \int_{-\pi}^{\pi} a_2 \cos 2t dt + \ldots$$
$$+ \int_{-\pi}^{\pi} b_1\sin t dt + \int_{-\pi}^{\pi} b_2\sin 2t dt + \ldots.$$

Since the integral of $\cos kt$ or $\sin kt$ over the interval from $-\pi$ to π vanishes, we conclude that

$$\int_{-\pi}^{\pi} f(t)dt = \pi a_0,$$

and we can solve for a_0, obtaining

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt. \tag{3.3}$$

To find the other Fourier coefficients, we will need some integral formulae. We claim that if m and n are positive integers,

$$\int_{-\pi}^{\pi} \cos nt \cos mt dt = \begin{cases} \pi, & \text{for } m = n, \\ 0, & \text{for } m \neq n, \end{cases} \tag{3.4}$$

$$\int_{-\pi}^{\pi} \sin nt \sin mt dt = \begin{cases} \pi, & \text{for } m = n, \\ 0, & \text{for } m \neq n, \end{cases} \tag{3.5}$$

$$\int_{-\pi}^{\pi} \sin nt \cos mt dt = 0. \tag{3.6}$$

Let us verify the first of these equations. We will use the trigonometric identities

$$\cos((n+m)t) = \cos nt \cos mt - \sin nt \sin mt,$$
$$\cos((n-m)t) = \cos nt \cos mt + \sin nt \sin mt.$$

Adding these together, we obtain

$$\cos((n+m)t) + \cos((n-m)t) = 2\cos nt \cos mt,$$

or

$$\cos nt \cos mt = \frac{1}{2}(\cos((n+m)t) + \cos((n-m)t)).$$

Hence

$$\int_{-\pi}^{\pi} \cos nt \cos mt dt = \frac{1}{2}\int_{-\pi}^{\pi} (\cos((n+m)t) + \cos((n-m)t))dt,$$

and since

$$\int_{-\pi}^{\pi} \cos(kt)dt = \frac{1}{k}\sin(kt)\Big|_{-\pi}^{\pi} = 0,$$

if $k \neq 0$, we conclude that

$$\int_{-\pi}^{\pi} \cos nt \cos mt dt = \begin{cases} \pi, & \text{for } m = n, \\ 0, & \text{for } m \neq n. \end{cases}$$

The reader is asked to verify the other two integral formulae (3.5) and (3.6) in one of the exercises at the end of this section.

To find the formula for the Fourier coefficients a_k for $k > 0$, we multiply both sides of (3.2) by $\sin kt$ and integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(t)\cos kt dt = \int_{-\pi}^{\pi} \frac{a_0}{2}\cos kt dt + \int_{-\pi}^{\pi} a_1 \cos t \cos kt dt + \ldots$$

$$+\ldots+\int_{-\pi}^{\pi} a_k \cos kt \cos kt dt + \ldots$$

$$+\ldots+\int_{-\pi}^{\pi} b_j \sin jt \cos kt dt + \ldots.$$

According to our formulae (3.4) and (3.6), there is only one term which survives:

$$\int_{-\pi}^{\pi} f(t) \cos kt dt = \int_{-\pi}^{\pi} a_k \cos kt \cos kt dt = \pi a_k.$$

We can easily solve for a_k:

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt dt. \tag{3.7}$$

A very similar argument yields the formula

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt dt. \tag{3.8}$$

Example. Let us use formulae (3.3), (3.7), and (3.8) to find the Fourier coefficients of the function

$$f(t) = \begin{cases} -\pi, & \text{for } -\pi < t < 0, \\ \pi, & \text{for } 0 < t < \pi, \\ 0, & \text{for } t = 0, \pi, \end{cases}$$

extended to be periodic of period 2π. Then

$$\frac{a_0}{2} = \text{average value of } f = 0,$$

and

$$a_m = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} f(t) \cos mt dt\right] = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos mt dt\right] + \frac{1}{\pi}\left[\int_{0}^{\pi} \pi \cos mt dt\right]$$

$$= \frac{1}{\pi}\frac{-\pi}{m}[\sin mt]_{-\pi}^{0} + \frac{1}{\pi}\frac{\pi}{m}[\sin mt]_{0}^{\pi} = \cdots = 0,$$

while

$$b_m = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} f(t) \sin mt dt\right] = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin mt dt\right] + \frac{1}{\pi}\left[\int_{0}^{\pi} \pi \sin mt dt\right]$$

$$= \frac{1}{\pi}\frac{\pi}{m}[\cos mt]_{-\pi}^{0} + \frac{1}{\pi}\frac{-\pi}{m}[\cos mt]_{0}^{\pi} = \frac{2}{m} - \frac{2}{m}\cos m\pi$$

$$= \frac{2}{m}(1-(-1)^m) = \begin{cases} \frac{4}{m}, & \text{if } m \text{ is odd}, \\ 0, & \text{if } m \text{ is even}. \end{cases}$$

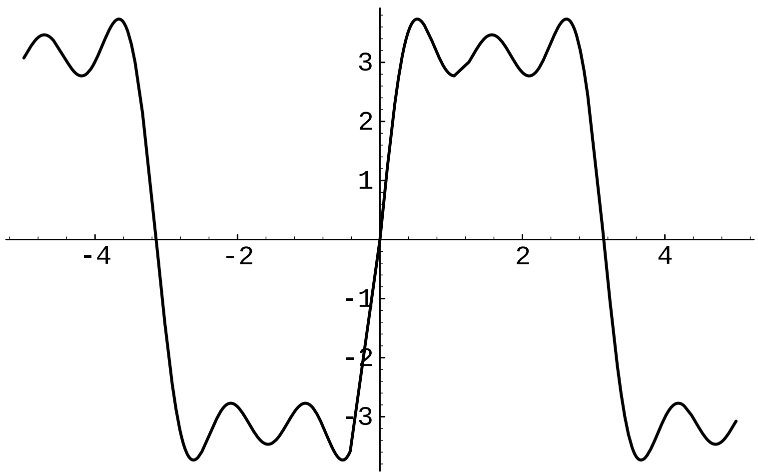

Figure 3.1: A graph of the Fourier approximation $\phi_5(t) = 4\sin t + (4/3)\sin 3t + (4/5)\sin 5t$.

Thus we find the Fourier series for f:

$$f(t) = 4\sin t + \frac{4}{3}\sin 3t + \frac{4}{5}\sin 5t + \cdots .$$

The trigonometric polynomials

$$\phi_1(t) = 4\sin t, \quad \phi_3(t) = 4\sin t + \frac{4}{3}\sin 3t$$

$$\phi_5(t) = 4\sin t + \frac{4}{3}\sin 3t + \frac{4}{5}\sin 5t$$

are approximations to $f(t)$ which improve as the number of terms increases.

By the way, this Fourier series yields a curious formula for π. If we set $t = \pi/2$, $f(t) = \pi$, and we obtain

$$\pi = 4\sin(\pi/2) + \frac{4}{3}\sin(3\pi/2) + \frac{4}{5}\sin(5\pi/2) + \cdots$$

from which we conclude that

$$\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots).$$

The Fourier series on the right-hand side of (3.1) is often conveniently expressed in the Σ notation,

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(\frac{2\pi kt}{T}) + \sum_{k=1}^{\infty} b_k \sin(\frac{2\pi kt}{T}), \tag{3.9}$$

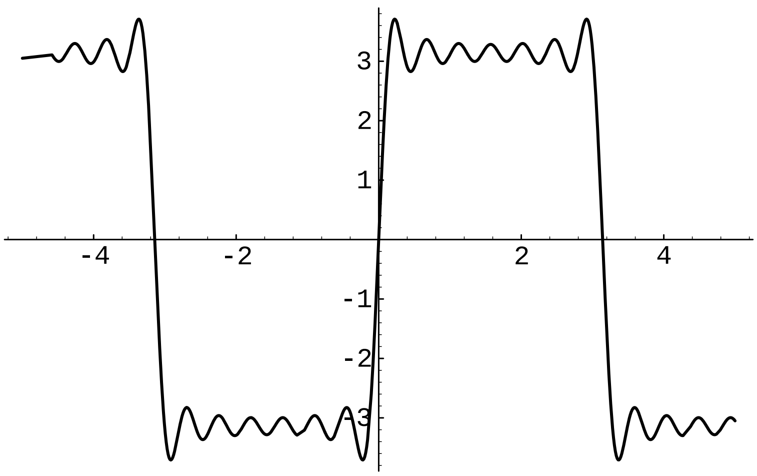

Figure 3.2: A graph of the Fourier approximation $\phi_{13}(t)$. The overshooting near the points of discontinuity is known as the "Gibbs phenomenon."

just as we did for power series in Chapter 1.

It is an interesting and difficult problem in harmonic analysis to determine how "well-behaved" a periodic function $f(t)$ must be in order to ensure that it can be expressed as a superposition of sines and cosines. An easily stated theorem, sufficient for many applications is:

Theorem 1. *If f is a continuous periodic function of period T, with continuous derivative f', then f can be written uniquely as a superposition of sines and cosines, in accordance with (3.9), where the a_k's and b_k's are constants. Moreover, the infinite series on the right-hand side of (3.9) converges to $f(t)$ for every choice of t.*

However, often one wants to apply the theory of Fourier series to functions which are not quite so well-behaved, in fact to functions that are not even continuous, such as the function in our previous example. A weaker sufficient condition for f to possess a Fourier series is that it be piecewise smooth.

The technical definition goes like this: A function $f(t)$ which is periodic of period T is said to be *piecewise smooth* if it is continuous and has a continuous derivative $f'(t)$ except at finitely many points of discontinuity within the interval $[0, T]$, and at each point t_0 of discontinuity, the right- and left-handed limits of f and f',

$$\lim_{t \to t_0+} (f(t)), \quad \lim_{t \to t_0-} (f(t)), \qquad \lim_{t \to t_0+} (f'(t)), \quad \lim_{t \to t_0-} (f'(t)),$$

all exist. The following theorem, proven in more advanced books,[2] ensures that a

[2]See, for example, Ruel Churchill and James Brown, *Fourier series and boundary value problems*, 4th edition, McGraw-Hill, New York, 1987 or Robert Seeley, *An introduction to Fourier series and integrals*, Benjamin, New York, 1966.

Fourier decomposition can be found for any function which is piecewise smooth. Although it applies to a more general class of functions than the preceding theorem, the convergence properties of the Fourier series one obtains are not quite as nice:

Theorem 2. *If f is any piecewise smooth periodic function of period T, f can be expressed as a Fourier series,*

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(\frac{2\pi kt}{T}) + \sum_{k=1}^{\infty} b_k \sin(\frac{2\pi kt}{T}) \tag{3.10}$$

where the a_k's and b_k's are constants. Here equality means that the infinite sum on the right converges to $f(t)$ for each t at which f is continuous. If f is discontinuous at t_0, its Fourier series at t_0 will converge to the average of the right and left hand limits of f as $t \to t_0$.

The formulae for the Fourier coefficients for functions of period T appearing in (3.10) (which generalize (3.7) and (3.8) for functions of period 2π) are

$$a_k = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(\frac{2\pi kt}{T}) dt \tag{3.11}$$

and

$$b_k = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(\frac{2\pi kt}{T}) dt. \tag{3.12}$$

Exercises:

3.1.1. The function $f(t) = \cos^2 t$ can be regarded as either periodic of period π or periodic of period 2π. Choose one of the two periods and find the Fourier series of $f(t)$. (Hint: This problem is very easy if you use trigonometric identities instead of trying to integrate directly.)

3.1.2. The function $f(t) = \sin^3 t$ is periodic of period 2π. Find its Fourier series. (Hint: Use trigonometric identities to calculate $\sin(3t) = \sin(t + 2t)$.)

3.1.3. The function

$$f(t) = \begin{cases} t, & \text{for } -\pi < t < \pi, \\ 0, & \text{for } t = \pi, \end{cases}$$

can be extended to be periodic of period 2π. Find the Fourier series of this extension.

3.1.4. The function

$$f(t) = |t|, \qquad \text{for } t \in [-\pi, \pi],$$

can be extended to be periodic of period 2π. Find the Fourier series of this extension.

3.1.5. Find the Fourier series of the following function:

$$f(t) = \begin{cases} t^2, & \text{for } -\pi \le t < \pi, \\ f(t - 2k\pi), & \text{for } -\pi + 2k\pi \le t < \pi + 2k\pi. \end{cases}$$

3.1.6. Establish the formulae (3.5) and (3.6), which were given in the text.

3.1.7. Using (3.11) and (3.12), find the Fourier series of the function

$$f(t) = |t|, \qquad \text{for } t \in [-1, 1],$$

which has been extended to be periodic of period 2.

3.2 Inner products

There is a convenient way of remembering the formulae for the Fourier coefficients that we derived in the preceding section which makes use of the notion of inner product. This concept also allows a deeper understanding as to why these formulae hold and provides insight into how they might be generalized in other contexts.

Mathematics often makes significant progress by means of analogy. Let V be the set of functions $f : \mathbb{R} \to \mathbb{R}$ which are *smooth*, that is, have continuous derivatives of arbitrary order, and are periodic of period 2π. (These functions satisfy the hypotheses of Theorem 1 from the previous section.) We want to think of elements of V as elements of infinite-dimensional Euclidean space, analogous to $\mathbb{R}^n$. Indeed, we could imagine approximating any element $f \in V$ by a vector

$$\left(f(0), f\left(\frac{2\pi}{n}\right), f\left(2\frac{2\pi}{n}\right), \dots, f\left((n-1)\frac{2\pi}{n}\right) \right) \in \mathbb{R}^n.$$

In fact, this is what must be done if we want to represent the function by a table of values that can be stored in a computer. (Any computer has insufficient memory to store the values of f at the infinitely many distinct points in the interval from 0 to 2π.)

We say that the space V of periodic functions is a *vector space* because elements of V can be added and multiplied by scalars, these operations satisfying the same rules as those for addition of ordinary vectors and multiplication of ordinary vectors by scalars. Indeed, the sum of functions f and g is just the function $f+g$ defined by $(f+g)(t) = f(t)+g(t)$, while the scalar product of the function f with a constant c is just the function cf defined by $(cf)(t) = cf(t)$. This may seem somewhat formal, but the idea really pays for itself in the long run.

We need an analog of the dot product, and for this purpose we define an *inner product* between elements of V by means of the formula

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t)dt.$$

Thus for example, if $f(t) = \sin t$ and $g(t) = 2\cos t$, then

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} 2 \sin t \cos t dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(2t) dt = -\frac{1}{\pi} \cos(2t)|_{-\pi}^{\pi} = 0.$$

This inner product has properties quite similar to those of the standard dot product on $\mathbb{R}^n$:

- $\langle f, g \rangle = \langle g, f \rangle$, whenever f and g are elements of V.
- $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$.
- $\langle cf, g \rangle = c\langle f, g \rangle$, when c is a real constant.
- $\langle f, f \rangle \geq 0$, with equality holding only if $f = 0$ (at all points of continuity).

This suggests that we might use geometric terminology for elements of V just as we did for vectors in $\mathbb{R}^n$. Thus, for example, we will say that an element f of V is of *unit length* if $\langle f, f \rangle = 1$ and that two elements f and g of V are *perpendicular* if $\langle f, g \rangle = 0$.

For example, let's consider the constant function $f(t) = 1/\sqrt{2}$. In this case,

$$\langle f, f \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{1}{\sqrt{2}} \right)^2 dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} dt = 1,$$

so this function has unit-length with respect to the inner product we have defined.

In this terminology, the formulae (3.4), (3.5), and (3.6) can be expressed by stating that the functions

$$1/\sqrt{2}, \quad \cos t, \quad \cos 2t, \quad \cos 3t, \quad \ldots, \quad \sin t, \quad \sin 2t, \quad \sin 3t, \quad \ldots$$

are of unit length and perpendicular to each other. Moreover, by the theorem in the preceding section, any element of of V can be written as a (possibly infinite) superposition of these functions. We will say that the functions

$$e_0(t) = \frac{1}{\sqrt{2}}, \quad e_1(t) = \cos t, \quad e_2(t) = \cos 2t,$$
$$\hat{e}_1(t) = \sin t, \quad \hat{e}_2(t) = \sin 2t, \quad \ldots$$

make up an *orthonormal basis* for V.

The notion of orthonormal basis is borrowed from vector algebra. You will recall that $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is a basis for R^3 consisting of vectors which are of unit-length and perpendicular to each other, and that

$$\mathbf{v} \in \mathbb{R}^3 \quad \Rightarrow \quad \mathbf{v} = (\mathbf{v} \cdot \mathbf{i})\mathbf{i} + (\mathbf{v} \cdot \mathbf{j})\mathbf{j} + (\mathbf{v} \cdot \mathbf{k})\mathbf{k}.$$

We say that $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is an orthonormal basis for R^3. More generally, an ordered n-tuple $\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_n$ of vectors in R^n which are of unit length and perpendicular to each other is called an orthonormal basis for R^n, an example being constructed from the vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ 0 \end{pmatrix}, \quad \cdots, \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ 1 \end{pmatrix}.$$

(Other examples could be constructed using rotated cartesian coordinate systems.) If $\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_n$ is an orthonormal basis for $\mathbb{R}^n$, then

$$\mathbf{f} \in \mathbb{R}^n \quad \Rightarrow \quad \mathbf{f} = (\mathbf{f} \cdot \mathbf{e}_1)\mathbf{e}_1 + \cdots + (\mathbf{f} \cdot \mathbf{e}_n)\mathbf{e}_n.$$

A similar formula holds when $\mathbb{R}^n$ is replaced by V and the dot product is replaced by the inner product $\langle \cdot, \cdot \rangle$ we have described, except that the formula has infinitely many terms: If f is any element of V, we can write

$$\begin{aligned} f(t) = \langle f(t), e_0(t) \rangle e_0(t) + \langle f(t), e_1(t) \rangle e_1(t) + \langle f(t), e_2(t) \rangle e_2(t) + \cdots \\ + \langle f(t), \hat{e}_1(t) \rangle \hat{e}_1(t) + \langle f(t), \hat{e}_2(t) \rangle \hat{e}_2(t) + \cdots . \end{aligned}$$

In other words,

$$\begin{aligned} f(t) &= \left\langle f, \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} + \langle f, \cos t \rangle \cos t + \langle f, \cos 2t \rangle \cos 2t + \ldots \\ &\qquad + \langle f, \sin t \rangle \sin t + \langle f, \sin 2t \rangle \sin 2t + \ldots \\ &= \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + \ldots + b_1 \sin t + b_2 \sin 2t + \ldots , \end{aligned}$$

where

$$\begin{aligned} a_1 = \langle f, \cos t \rangle, \quad a_2 = \langle f, \cos 2t \rangle, \quad \ldots , \\ b_1 = \langle f, \sin t \rangle, \quad b_2 = \langle f, \sin 2t \rangle, \quad \ldots , \end{aligned}$$

and

$$\frac{a_0}{2} = \left\langle f, \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} = \frac{1}{2} \langle f, 1 \rangle, \quad \text{so} \quad a_0 = \langle f, 1 \rangle.$$

Use of the inner product makes the formulae for Fourier coefficients easier to remember.

We can define a differential operator

$$\frac{d^2}{dt^2} : V \to V, \quad \text{defined by} \quad \left(\frac{d^2}{dt^2}(f) \right)(t) = \frac{d^2 f}{dt^2}(t).$$

This operator is said to be linear because

$$\frac{d^2}{dt^2}(f + g) = \frac{d^2}{dt^2}(f) + \frac{d^2}{dt^2}(g) \quad \text{and} \quad \frac{d^2}{dt^2}(cf) = c \frac{d^2}{dt^2}(f),$$

when c is a constant. Moreover, it takes the sine and cosine functions to constant multiples of themselves:

$$\frac{d^2}{dt^2}(\sin(nt)) = \frac{d}{dt}(n\cos(nt)) = -n^2\sin(nt),$$
$$\frac{d^2}{dt^2}(\cos(nt)) = \frac{d}{dt}(-n\sin(nt)) = -n^2\cos(nt).$$

Thus the functions $\sin(nt)$ and $\cos(nt)$ satisfy the equation

$$\frac{d^2}{dt^2}(f) = \lambda f, \quad \text{with} \quad \lambda = -n^2.$$

We therefore say that $\sin(nt)$ and $\cos(nt)$ are *eigenfunctions* for the operator (d^2/dt^2) with *eigenvalue* $-n^2$.

Remark. Sometimes it is convenient to define a function $f : R \to R$ by means of a Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt),$$
$$\text{in which the infinite sum} \quad \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \quad \text{converges.}$$

(The condition on convergence is imposed so that the inner product $\langle f, f\rangle$ will be a finite number.) The collection H of functions that can be described in this way is larger than the space V of functions which have continuous derivatives of all orders. The space H is sometimes called the *completion* of V or *Hilbert space*, in honor of the celebrated German mathematician David Hilbert, who made many of the fundamental contributions to the theory presented in these pages.

Even and odd functions. We say that a function $f(t)$ is *even* if $f(-t) = f(t)$ or *odd* if $f(-t) = -f(t)$. One can check that

$$1, \qquad \cos t, \qquad \cos 2t, \qquad \cos 3t, \qquad \ldots$$

are even functions, while

$$\sin t, \qquad \sin 2t, \qquad \sin 3t, \qquad \ldots$$

are odd functions. Let

$$W_{\text{even}} = \{f \in V : f \text{ is even}\}, \qquad W_{\text{odd}} = \{f \in V : f \text{ is odd}\}.$$

Then

$$f, g \in W_{\text{even}} \qquad \Rightarrow \qquad f + g \in W_{\text{even}} \qquad \text{and} \qquad cf \in W_{\text{even}},$$

for every choice of scalar c. In the language of linear algebra, we can say that W_{even} is a linear subspace of V. Similarly, W_{odd} is a linear subspace of V.

It is not difficult to show that

$$f \in W_{\text{even}}, \qquad g \in W_{\text{odd}} \qquad \Rightarrow \qquad \langle f, g \rangle = 0;$$

in other words, the linear subspaces W_{even} and W_{odd} are perpendicular to each other with respect to the inner produce we defined. Indeed, under these conditions, fg is odd and hence

$$\begin{aligned}
\langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt &= \frac{1}{\pi}\int_{-\pi}^{0} f(t)g(t)dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)g(t)dt \\
&= \frac{1}{\pi}\int_{\pi}^{0} f(-t)g(-t)(-dt) + \frac{1}{\pi}\int_{0}^{\pi} f(t)g(t)dt \\
&= -\frac{1}{\pi}\int_{0}^{\pi} -(f(t)g(t))(-dt) + \frac{1}{\pi}\int_{0}^{\pi} f(t)g(t)dt = 0.
\end{aligned}$$

The variable of integration has been changed from t to $-t$ in the first integral of the second line.

It follows that if $f \in W_{\text{even}}$,

$$b_n = \langle f, \sin nt \rangle = 0, \quad \text{for } n > 0.$$

Similarly, if $f \in W_{\text{odd}}$,

$$a_0 = \langle f, 1 \rangle = 0, \qquad a_n = \langle f, \cos nt \rangle = 0, \quad \text{for } n > 0.$$

Thus for an even or odd function, half of the Fourier coefficients are automatically zero. This simple fact can often simplify the calculation of Fourier coefficients.

Moreover, in the case where f is even, we find that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(t)\cos(nt)dt + \int_{0}^{\pi} f(t)\cos(nt)dt\right].$$

Making the substitution $u = -t$ in the first integral yields

$$\begin{aligned}
a_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(-u)\cos(-nu)(-du) + \int_{0}^{\pi} f(t)\cos(nt)dt\right] \\
&= \frac{1}{\pi}\left[\int_{0}^{\pi} f(u)\cos(nu)(du) + \int_{0}^{\pi} f(t)\cos(nt)dt\right] = \frac{2}{\pi}\int_{0}^{\pi} f(t)\cos(nt)dt.
\end{aligned}$$

Similarly, if f is odd, we find that

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(t)\sin(nt)dt.$$

Exercises:

3.2.1. Evaluate the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) g(t) dt,$$

in the case where $f(t) = \cos 2t$ and $g(t) = |\sin t|$.

3.2.2. We could define an inner product on functions $f : [1, e^{\pi}] \to \mathbb{R}$ by the formula

$$\langle f, g \rangle = \int_{1}^{e^{\pi}} \frac{1}{t} f(t) g(t) dt.$$

Evaluate this inner product in the case where $f(t) = \cos(\log(t))$ and $g(t) = 1$, where log denotes the natural or base e logarithm. (Hint: Use the substitution $t = e^u$.)

3.2.3. Determine which of the following functions are even, which are odd, and which are neither even nor odd:

a. $f(t) = t^3 + 3t$.

b. $f(t) = t^2 + |t|$.

c. $f(t) = e^t$.

d. $f(t) = \frac{1}{2}(e^t + e^{-t})$.

e. $f(t) = \frac{1}{2}(e^t - e^{-t})$.

f. $f(t) = J_0(t)$, the Bessel function of the first kind.

3.2.4. The function

$$f(t) = \begin{cases} 0 & \text{for } -\pi \leq t < -\pi/2 \\ -1 & \text{for } -\pi/2 \leq t < 0 \\ 0 & \text{for } t = 0 \\ 1 & \text{for } 0 \leq t \leq \pi/2 \\ 0 & \text{for } \pi/2 < t \leq \pi \end{cases}$$

can be extended to be periodic of period 2π.

a. Is the extended function even, odd, or neither even nor odd?

b. Find the Fourier series of the extended function.

3.3 Fourier sine and cosine series

Let $f : [0, L] \to \mathbb{R}$ be a continuously differentiable function which vanishes at 0 and L. We claim that we can express $f(t)$ as the superposition of sine functions,

$$f(t) = b_1 \sin(\pi t/L) + b_2 \sin(2\pi t/L) + \ldots + b_n \sin(n\pi t/L) + \ldots,$$

where $b_1, b_2, \ldots$ are constants, which can also be written more succinctly in sigma notation as

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t/L). \tag{3.13}$$

We could prove this using the theory of even and odd functions. Indeed, we can extend f to an odd function $\tilde{f} : [-L, L] \to \mathbb{R}$ by setting

$$\tilde{f}(t) = \begin{cases} f(t), & \text{for } t \in [0, L], \\ -f(-t), & \text{for } t \in [-L, 0], \end{cases}$$

then to a function $\hat{f} : \mathbb{R} \to \mathbb{R}$, which is periodic of period $2L$ by requiring that

$$\hat{f}(t + 2L) = \hat{f}(t), \qquad \text{for all } t \in \mathbb{R}.$$

The extended function lies in the linear subspace W_{odd}. It follows from the calculations in Section 3.1 that $\hat{f}$ possesses a Fourier series expansion, and from the fact that $\hat{f}$ is odd that all of the a_n's are zero. On the interval $[0, L]$, $\hat{f}$ restricts to f and the Fourier expansion of $\hat{f}$ restricts to an expansion of f of the form (3.13) which involves only sine functions. We call (3.13) the *Fourier sine series* of f.

A similar argument can be used to express a continuously differentiable function $f : [0, L] \to \mathbb{R}$ into a superposition of cosine functions,

$$f(t) = \frac{a_0}{2} + a_1 \cos(\pi t/L) + a_2 \cos(2\pi t/L) + \ldots + a_n \cos(n\pi t/L) + \ldots. \tag{3.14}$$

To obtain this expression, we first extend f to an even function $\tilde{f} : [-L, L] \to \mathbb{R}$ by setting

$$\tilde{f}(t) = \begin{cases} f(t), & \text{for } t \in [0, L], \\ f(-t), & \text{for } t \in [-L, 0], \end{cases}$$

then to a function $\hat{f} : \mathbb{R} \to \mathbb{R}$ which is periodic of period $2L$, by requiring that

$$\hat{f}(t + 2L) = \hat{f}(t), \qquad \text{for all } t \in \mathbb{R}.$$

This time the extended function lies in the linear subspace W_{even}. It follows from the calculations in Section 3.1 that $\hat{f}$ possesses a Fourier series expansion, and from the fact that $\hat{f}$ is even that all of the b_n's are zero. On the interval $[0, L]$, $\hat{f}$ restricts to f and the Fourier expansion of $\hat{f}$ restricts to an expansion of f of the form (3.14) which involves only cosine functions. We call (3.14) the *Fourier cosine series* of f.

Formulae for the coefficients of the Fourier sine and cosine series were actually given at the end of the previous section in the case where $L = \pi$. However, we can also generate these formulae for arbitrary L by defining slightly different inner products than the one considered in the preceding section. This time, we let V be the vector space of functions $f : [0, L] \to \mathbb{R}$ with continuous derivatives of all orders, and let

$$V_0 = \{f \in V : f(0) = 0 = f(L)\},$$

a linear subspace of V. We define an inner product $\langle \cdot, \cdot \rangle$ on the infinite-dimensional vector space V_0 by means of the formula

$$\langle f, g \rangle = \frac{2}{L} \int_0^L f(t)g(t)dt.$$

This restricts to an inner product on V_0.

Let's consider now the Fourier sine series. We have seen that any element of V_0 can be represented as a superpostion of the sine functions

$$\sin(\pi t/L), \quad \sin(2\pi t/L), \quad \ldots, \quad \sin(n\pi t/L), \quad \ldots.$$

We claim that these sine functions form an orthonormal basis for V_0 with respect to the inner product we have defined. Recall the trigonometric formulae that we used in §3.1:

$$\cos((n+m)\pi t/L) = \cos(n\pi t/L)\cos(m\pi t/L) - \sin(n\pi t/L)\sin(m\pi t/L),$$

$$\cos((n-m)\pi t/L) = \cos(n\pi t/L)\cos(m\pi t/L) + \sin(n\pi t/L)\sin(m\pi t/L).$$

Subtracting the first of these from the second and dividing by two yields

$$\sin(n\pi t/L)\sin(m\pi t/L) = \frac{1}{2}(\cos((n-m)\pi t/L) - \cos((n+m)\pi t/L),$$

and hence

$$\int_0^L \sin(n\pi t/L)\sin(m\pi t/L)dt =$$

$$\frac{1}{2}\int_0^L (\cos((n-m)\pi t/L) - \cos((n+m)\pi t/L)dt.$$

If n and m are positive integers, the integral on the right vanishes unless $n = m$, in which case the right-hand side becomes

$$\frac{1}{2}\int_0^L dt = \frac{L}{2}.$$

Hence

$$\frac{2}{L}\int_0^L \sin(n\pi t/L)\sin(m\pi t/L)dt = \begin{cases} 1, & \text{for } m = n, \\ 0, & \text{for } m \neq n, \end{cases}$$

which can be expressed in terms of our inner product as

$$\langle \sin(n\pi t/L), \sin(m\pi t/L) \rangle = \begin{cases} 1, & \text{for } m = n, \\ 0, & \text{for } m \neq n. \end{cases} \tag{3.15}$$

This just says that the functions

$$\sin(\pi t/L), \quad \sin(2\pi t/L), \quad \ldots, \quad \sin(n\pi t/L), \quad \ldots$$

form an othonormal basis for the infinite-dimensional vector space V_0. Therefore, just as in the previous section, we can evaluate the coefficients of the Fourier sine series of a function $f \in V_0$ by simply projecting f onto each element of this orthonormal basis. When we do this, we find that

$$\begin{aligned} f(t) = b_1 \sin(\pi t/L) + b_2 \sin(2\pi t/L) + \ldots + b_n \sin(n\pi t/L) + \ldots \\ = \sum_{n=1}^{\infty} b_n \sin(n\pi t/L), \end{aligned}$$

where

$$b_n = \langle f, \sin(n\pi t/L)\rangle = \frac{2}{L}\int_0^L f(t)\sin(n\pi t/L)dt. \tag{3.16}$$

We can treat the Fourier cosine series in a similar fashion. In this case, we let

$$\begin{aligned} V_1 = \{f : [0, L] \to \mathbb{R} : \ f \text{ has continuous} \\ \text{derivatives of all orders and } f'(0) = 0 = f'(L)\ \}, \end{aligned}$$

and show that the functions

$$\frac{1}{\sqrt{2}}, \quad \cos(\pi t/L), \quad \cos(2\pi t/L), \quad \ldots, \quad \cos(n\pi t/L), \quad \ldots$$

form an orthonormal basis for V_1. Thus we can evaluate the coefficients of the Fourier cosine series of a function $f \in V_1$ by projecting f onto each element of this orthonormal basis. We will leave it to the reader to carry this out in detail, and simply remark that when the dust settles, one obtains the following formula for the coefficient a_n in the Fourier cosine series:

$$a_n = \frac{2}{L}\int_0^L f(t)\cos(n\pi t/L)dt. \tag{3.17}$$

Just as for ordinary Fourier series, we can take Fourier sine and cosine series for functions $f : [0, L] \to \mathbb{R}$ which are only piecewise smooth. Such functions need not satisfy the endpoint conditions we used in the definitions of V_0 and V_1.

Example. First let us use (3.16) to find the Fourier sine series of

$$f(t) = \begin{cases} t, & \text{for } 0 \le t \le \pi/2, \\ \pi - t, & \text{for } \pi/2 \le t \le \pi. \end{cases} \tag{3.18}$$

In this case, $L = \pi$, and according to our formula,

$$b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} t \sin nt dt + \int_{\pi/2}^{\pi} (\pi - t)\sin nt dt\right].$$

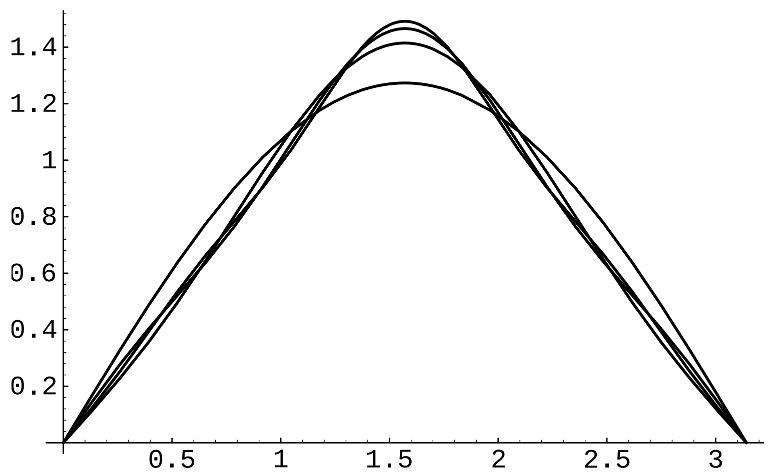

Figure 3.3: A graph of the Fourier sine approximations ϕ_1, ϕ_3, ϕ_5 and ϕ_7.

We use integration by parts to obtain

$$\int_0^{\pi/2} t \sin nt dt = \left[\frac{-t}{n} \cos nt\right]\Big|_0^{\pi/2} + \int_0^{\pi/2} \frac{1}{n} \cos nt dt$$

$$= \frac{-\pi \cos(n\pi/2)}{2n} + \frac{1}{n^2}[\sin nt]|_0^{\pi/2} = \frac{-\pi \cos(n\pi/2)}{2n} + \frac{\sin(n\pi/2)}{n^2},$$

while

$$\int_{\pi/2}^{\pi} (\pi - t) \sin nt dt = \left[\frac{-(\pi - t)}{n} \cos(nt)\right]\Big|_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \frac{1}{n} \cos nt dt$$

$$= \frac{\pi \cos(n\pi/2)}{2n} - \frac{1}{n^2}[\sin nt]|_{\pi/2}^{\pi} = \frac{\pi \cos(n\pi/2)}{2n} + \frac{\sin(n\pi/2)}{n^2}.$$

Thus

$$b_n = \frac{4 \sin(n\pi/2)}{\pi n^2},$$

and the Fourier sine series of $f(t)$ is

$$f(t) = \frac{4}{\pi} \sin t - \frac{4}{9\pi} \sin 3t + \frac{4}{25\pi} \sin 5t - \frac{4}{49\pi} \sin 7t + \ldots. \tag{3.19}$$

The trigonometric polynomials

$$\phi_1(t) = \frac{4}{\pi} \sin t, \quad \phi_3(t) = \frac{4}{\pi} \sin t - \frac{4}{9\pi} \sin 3t, \ldots$$

are better and better approximations to the function $f(t)$.

To find the Fourier cosine series of

$$f(t) = \begin{cases} t, & \text{for } 0 \le t \le \pi/2, \\ \pi - t, & \text{for } \pi/2 \le t \le \pi, \end{cases}$$

we first note that

$$a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} t dt + \int_{\pi/2}^{\pi} (\pi - t) dt\right] = \frac{2}{\pi}\left[\frac{1}{2}\left(\frac{\pi}{2}\right)^2 + \frac{1}{2}\left(\frac{\pi}{2}\right)^2\right] = \frac{\pi}{2}.$$

To find the other a_n's, we use (3.17):

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} t \cos ntdt + \int_{\pi/2}^{\pi} (\pi - t) \cos ntdt\right].$$

This time, integration by parts yields

$$\begin{aligned} \int_0^{\pi/2} t \cos ntdt &= \left[\frac{t}{n} \sin nt\right]\Big|_0^{\pi/2} - \int_0^{\pi/2} \frac{1}{n} \sin ntdt \\ &= \frac{\pi \sin(n\pi/2)}{2n} + \frac{1}{n^2}[\cos nt]|_0^{\pi/2} \\ &= \frac{\pi \sin(n\pi/2)}{2n} + \frac{\cos(n\pi/2)}{n^2} - \frac{1}{n^2} \end{aligned}$$

while

$$\begin{aligned} \int_{\pi/2}^{\pi} (\pi - t) \cos ntdt &= \left[\frac{(\pi - t)}{n} \sin(nt)\right]\Big|_{\pi/2}^{\pi} + \int_{\pi/2}^{\pi} \frac{1}{n} \sin ntdt \\ &= \frac{-\pi \sin(n\pi/2)}{2n} - \frac{1}{n^2}[\cos nt]|_{\pi/2}^{\pi} \\ &= \frac{-\pi \sin(n\pi/2)}{2n} + \frac{\cos(n\pi/2)}{n^2} - \frac{1}{n^2}\cos(n\pi). \end{aligned}$$

Thus when $n \ge 1$,

$$a_n = \frac{2}{\pi n^2}[2\cos(n\pi/2) - 1 - 1(-1)^n],$$

and the Fourier sine series of $f(t)$ is

$$f(t) = \frac{\pi}{4} - \frac{2}{\pi}\cos 2t - \frac{2}{9\pi}\cos 6t - \frac{2}{25\pi}\cos 10t - \dots . \tag{3.20}$$

Note that we have expanded exactly the same function $f(t)$ on the interval $[0, \pi]$ as either a superposition of sines in (3.19) or as a superposition of cosines in (3.20).

Exercises:

3.3.1.a. Find the Fourier sine series of the function $f : [0, \pi] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x < \pi/2, \\ 0 & \text{for } \pi/2 \leq x \leq \pi. \end{cases}$$

b. Find the Fourier cosine series of the same function.

3.3.2.a. Find the Fourier sine series of the following function defined on the interval $[0, \pi]$:

$$f(t) = t(\pi - t).$$

b. Find the Fourier cosine series of the same function.

3.3.3. Find the Fourier sine series of the following function defined on the interval $[0, 10]$:

$$f(t) = \begin{cases} t, & \text{for } 0 \leq t < 5, \\ 10 - t, & \text{for } 5 \leq t \leq 10. \end{cases}$$

3.3.4. Find the Fourier sine series of the following function defined on the interval $[0, 1]$:

$$f(t) = 5t(1 - t).$$

3.3.5.(For students with access to Mathematica) Find numerical approximations to the first ten coefficients of the Fourier sine series for the function

$$f(t) = t + t^2 - 2t^3,$$

defined for t in the interval $[0, 1]$, by running the following Mathematica program

```
f[n_] := 2 NIntegrate[(t + t∧2 - 2 t∧3) Sin[n Pi t], {t,0,1}];
b = Table[f[n], {n,1,10}]
```

3.4 Complex version of Fourier series*

We have seen that if $f : R \to R$ is a well-behaved function which is periodic of period 2π, f can be expanded in a Fourier series

$$f(t) = \frac{a_0}{2} + a_1 \cos t + a_2 \cos(2t) + \dots$$
$$+ b_1 \sin t + b_2 \sin(2t) + \dots .$$

We say that this is the *real form* of the Fourier series. It is often convenient to recast this Fourier series in *complex form* by means of the Euler formula, which states that

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

It follows from this formula that

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta, \qquad e^{-i\theta} + e^{-i\theta} = 2i\sin\theta,$$

or

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} + e^{-i\theta}}{2i}.$$

Hence the Fourier expansion of f can be rewritten as

$$f(t) = \frac{a_0}{2} + \frac{a_1}{2}(e^{it} + e^{-it}) + \frac{a_2}{2}(e^{2it} + e^{-2it}) + \dots$$

$$+\frac{b_1}{2i}(e^{it} - e^{-it}) + \frac{b_2}{2i}(e^{2it} - e^{-2it}) + \dots,$$

or

$$f(t) = \dots + c_{-2}e^{-2it} + c_{-1}e^{-it} + c_0 + c_1 e^{it} + c_2 e^{2it} + \dots, \tag{3.21}$$

where the c_k's are the complex numbers defined by the formulae

$$c_0 = \frac{a_0}{2}, \quad c_1 = \frac{a_1 - ib_1}{2}, \quad c_2 = \frac{a_2 - ib_2}{2}, \quad \dots,$$

$$c_{-1} = \frac{a_1 + ib_1}{2}, \quad c_{-2} = \frac{a_2 + ib_2}{2}, \quad \dots.$$

If $k \neq 0$, we can solve for a_k and b_k:

$$a_k = c_k + c_{-k}, \qquad b_k = i(c_k - c_{-k}). \tag{3.22}$$

It is not difficult to check the following integral formula via direct integration:

$$\int_{-\pi}^{\pi} e^{imt}e^{-int}dt = \begin{cases} 2\pi, & \text{for } m = n, \\ 0 & \text{for } m \neq n. \end{cases} \tag{3.23}$$

If we multiply both sides of (3.21) by e^{-ikt}, integrate from $-\pi$ to π and apply (3.23), we obtain

$$\int_{-\pi}^{\pi} f(t)e^{-ikt}dt = 2\pi c_k,$$

which yields the formula for coefficients of the complex form of the Fourier series:

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dt. \tag{3.24}$$

Example. Let us use formula (3.24) to find the complex Fourier coefficients of the function

$$f(t) = t \qquad \text{for } -\pi < t \leq \pi,$$

extended to be periodic of period 2π. Then

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} te^{-ikt}dt.$$

We apply integration by parts with $u = t$, $dv = e^{-ikt}dt$, $du = dt$ and $v = (i/k)e^{-ikt}$:

$$c_k = \frac{1}{2\pi}\left[(it/k)e^{-ikt}|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(i/k)e^{-ikt}dt\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi i}{k}e^{-i\pi k} + \frac{\pi i}{k}e^{i\pi k}\right] = i\frac{(-1)^k}{k}.$$

It follows from (3.22) that

$$a_k = (c_k + c_{-k}) = 0, \qquad b_k = i(c_k - c_{-k}) = -2\frac{(-1)^k}{k}.$$

It is often the case that the complex form of the Fourier series is far simpler to calculate than the real form. One can then use (3.22) to find the real form of the Fourier series.

Exercise:

3.4.1. Prove the integral formula (3.23), presented in the text.

3.4.2. a. Find the complex Fourier coefficients of the function

$$f(t) = t^2 \qquad \text{for } -\pi < t \leq \pi,$$

extended to be periodic of period 2π.

b. Use (3.22) to find the real form of the Fourier series.

3.4.3. a. Find the complex Fourier coefficients of the function

$$f(t) = t(\pi - t) \qquad \text{for } -\pi < t \leq \pi,$$

extended to be periodic of period 2π.

b. Use (3.22) to find the real form of the Fourier series.

3.4.4. Show that if a function $f : R \to R$ is smooth and periodic of period $2\pi L$, we can write the Fourier expansion of f as

$$f(t) = \ldots + c_{-2}e^{-2it/L} + c_{-1}e^{-it/L} + c_0 + c_1 e^{it/L} + c_2 e^{2it/L} + \ldots,$$

where

$$c_k = \frac{1}{2\pi L}\int_{-\pi L}^{\pi L} f(t)e^{-ikt/L}dt.$$

3.5 Fourier transforms*

One of the problems with the theory of Fourier series presented in the previous sections is that it applies only to periodic functions. There are many times when one would like to divide a function which is not periodic into a superposition of sines and cosines. The Fourier transform is the tool often used for this purpose.

The idea behind the Fourier transform is to think of $f(t)$ as vanishing outside a very long interval $[-\pi L, \pi L]$. The function can be extended to a periodic function $f(t)$ such that $f(t + 2\pi L) = f(t)$. According to the theory of Fourier series in complex form (see Exercise 3.4.4),

$$f(t) = \ldots + c_{-2}e^{-2it/L} + c_{-1}e^{-it/L} + c_0 + c_1 e^{it/L} + c_2 e^{2it/L} + \ldots,$$

where the c_k's are the complex numbers.

Definition. If $f : \mathbb{R} \to \mathbb{R}$ is a piecewise continuous function which vanishes outside some finite interval, its *Fourier transform* is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{-i\xi t}dt. \tag{3.25}$$

The integral in this formula is said to be *improper* because it is taken from $-\infty$ to ∞; it is best to regard it as a limit,

$$\int_{-\infty}^{\infty} f(t)e^{-i\xi t}dt = \lim_{L\to\infty} \int_{-\pi L}^{\pi L} f(t)e^{-i\xi t}dt.$$

To explain (3.25), we suppose that f vanishes outside the interval $[-\pi L, \pi L]$. We can extend the restriction of f to this interval to a function which is periodic of period $2\pi L$. Then

$$\hat{f}(k/L) = \int_{-\infty}^{\infty} f(t)e^{-ikt/L}dt = \int_{-\pi L}^{\pi L} \tilde{f}(t)e^{-ikt/L}dt$$

represents $2\pi L$ times the Fourier coefficient of this extension of frequency k/L; indeed, it follows from (3.24) that we can write

$$f(t) = \ldots + c_{-2}e^{-2it/L} + c_{-1}e^{-it/L} + c_0 + c_1 e^{it/L} + c_2 e^{2it/L} + \ldots,$$

for $t \in [-\pi L, \pi L]$, where

$$c_k = \frac{1}{2\pi L}\hat{f}(k/L),$$

or alternatively,

$$f(t) = \ldots + \frac{1}{2\pi L}\hat{f}(-2/L)e^{-2it/L} + \frac{1}{2\pi L}\hat{f}(-1/L)e^{-it/L} +$$
$$+ \frac{1}{2\pi L}\hat{f}(0) + \frac{1}{2\pi L}\hat{f}(1/L)e^{it/L} + \frac{1}{2\pi L}\hat{f}(2/L)e^{2it/L} + \ldots.$$

In the limit as $L \to \infty$, it can be shown that this last sum approaches an improper integral, and our formula becomes

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi t}d\xi. \tag{3.26}$$

Equation (3.26) is called the *Fourier inversion formula*. If we make use of Euler's formula, we can write the Fourier inversion formula in terms of sines and cosines,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\cos\xi t d\xi + \frac{i}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\sin\xi t d\xi,$$

a superposition of sines and cosines of various frequencies.

Equations (3.25) and (3.25) allow one to pass back and forth between a given function and its representation as a superposition of oscillations of various frequencies. Like the Laplace transform, the Fourier transform is often an effective tool in finding explicit solutions to differential equations.

Exercise:

3.5.1. Find the Fourier transform of the function $f(t)$ defined by

$$f(t) = \begin{cases} 1, & \text{if } -1 \leq t \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Chapter 4

Partial Differential Equations

4.1 Overview

A *partial differential equation* is an equation which contains partial derivatives, such as the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

in which u is regarded as a function of x and t. Unlike the theory of ordinary differential equations which centers upon one key theorem—the fundamental existence and uniqueness theorem—there is no real unified theory of partial differential equations. Instead, each type of partial differential equations exhibits its own special features, which usually mirror the physical phenomena which the equation was first used to model.

Many of the foundational theories of physics and engineering are expressed by means of systems of partial differential equations. The reader may have heard some of these equations mentioned in previous courses in physics. Fluid mechanics is often formulated by the Euler equations of motion or the so-called Navier-Stokes equations, electricity and magnetism by Maxwell's equations, general relativity by Einstein's field equations. It is therefore important to develop techniques that can be used to solve a wide variety of partial differential equations.

In this chapter, we will give two important simple examples of partial differential equations, the heat equation and the wave equation, and we will show how to solve them by the techniques of "separation of variables" and Fourier analysis. Higher dimensional examples will be given in the following chapter. We will see that just as in the case of ordinary differential equations, there is an important dichotomy between linear and nonlinear equations. The techniques of separation of variables and Fourier analysis we present here are effective only for linear partial differential equations. Nonlinear partial differential equations are far more difficult to solve, and form a key topic of contemporary mathematical research.[1]

Our first example is the equation governing propagation of heat in a bar of length L. We imagine that the bar is located along the x-axis and we let

$$u(x,t) = \text{temperature of the bar at the point } x \text{ at time } t.$$

Heat in a small segment of a homogeneous bar is proportional to temperature, the constant of proportionality being determined by the density and specific heat of the material making up the bar. More generally, if $\sigma(x)$ denotes the specific heat at the point x and $\rho(x)$ is the density of the bar at x, then the heat within the region D_{x_1,x_2} between x_1 and x_2 is given by the formula

$$\text{Heat within } D_{x_1,x_2} = \int_{x_1}^{x_2} \rho(x)\sigma(x)u(x,t)dx.$$

To calculate the rate of change of heat within D_{x_1,x_2} with respect to time, we simply differentiate under the integral sign:

$$\frac{d}{dt}\left[\int_{x_1}^{x_2} \rho(x)\sigma(x)u(x,t)dx\right] = \int_{x_1}^{x_2} \rho(x)\sigma(x)\frac{\partial u}{\partial t}(x,t)dx.$$

Now heat is a form of energy, and conservation of energy implies that if no heat is being created or destroyed in D_{x_1,x_2}, the rate of change of heat within D_{x_1,x_2} is simply the rate at which heat enters D_{x_1,x_2}. Hence the rate at which heat leaves D_{x_1,x_2} is given by the expression

$$\text{Rate at which heat leaves } D_{x_1,x_2} = -\int_{x_1}^{x_2} \rho(x)\sigma(x)\frac{\partial u}{\partial t}(x,t)dx. \tag{4.1}$$

(More generally, if heat is being created within D_{x_1,x_2}, say by a chemical reaction, at the rate $\mu(x)u(x,t)+\nu(x)$ per unit volume, then the rate at which heat leaves D_{x_1,x_2} is

$$-\int_{x_1}^{x_2} \rho(x)\sigma(x)\frac{\partial u}{\partial t}(x,t)dx + \int_{x_1}^{x_2} (\mu(x)u(x,t)+\nu(x))dx.)$$

On the other hand, the rate of heat flow $F(x,t)$ is proportional to the partial derivative of temperature,

$$F(x,t) = -\kappa(x)\frac{\partial u}{\partial x}(x,t), \tag{4.2}$$

[1] Further reading can be found in the many excellent upper-division texts on partial differential equations. We especially recommend Mark Pinsky, *Partial differential equations and boundary-value problems with applications*, 2nd edition, McGraw-Hill, 1991. An excellent but much more advanced book is Michael Taylor, *Partial differential equations: basic theory*, Springer, New York, 1996.

where $\kappa(x)$ is the thermal conductivity of the bar at x. Thus we find that the rate at which heat leaves the region D_{x_1,x_2} is also given by the formula

$$F(x_2,t) - F(x_1,t) = \int_{x_1}^{x_2} \frac{\partial F}{\partial x}(x,t)dx. \tag{4.3}$$

Comparing the two formulae (4.1) and (4.3), we find that

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial x}(x,t)dx = -\int_{x_1}^{x_2} \rho(x)\sigma(x)\frac{\partial u}{\partial t}(x,t)dx.$$

This equation is true for *all* choices of x_1 and x_2, so the integrands on the two sides must be equal:

$$\frac{\partial F}{\partial x} = -\rho(x)\sigma(x)\frac{\partial u}{\partial t}.$$

It follows from (4.2) that

$$\frac{\partial}{\partial x}\left(-\kappa(x)\frac{\partial u}{\partial x}\right) = -\rho(x)\sigma(x)\frac{\partial u}{\partial t}.$$

In this way, we obtain the *heat equation*

$$\rho(x)\sigma(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u}{\partial x}\right). \tag{4.4}$$

In the more general case in which heat is being created at the rate $\mu(x)u(x,t) + \nu(x)$ per unit length, one could show that heat flow is modeled by the equation

$$\rho(x)\sigma(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u}{\partial x}\right) + \mu(x)u + \nu(x). \tag{4.5}$$

In the special case where the bar is *homogeneous*, i.e. its properties are the same at every point, $\rho(x)$, $\sigma(x)$ and $\kappa(x)$ are constants, say σ and κ respectively, and (4.4) becomes

$$\frac{\partial u}{\partial t} = \frac{\kappa}{\rho\sigma}\frac{\partial^2 u}{\partial x^2}. \tag{4.6}$$

This is our simplest example of a linear partial differential equation. Although its most basic application concerns diffusion of heat, it arises in many other contexts as well. For example, a slight modification of the heat equation was used by Black and Scholes to price derivatives in financial markets.[2]

Exercises:

4.1.1. Study of heat flow often leads to "boundary-value problems" for ordinary differential equations. Indeed, in the "steady-state" case, in which u is independent of time, equation (4.5) becomes

$$\frac{d}{dx}\left(\kappa(x)\frac{du}{dx}(x)\right) + \mu(x)u(x) + \nu(x) = 0,$$

[2]A description of the Black-Scholes technique for pricing puts and calls is given in Paul Wilmott, Sam Howison and Jeff Dewynne, *The mathematics of financial derivatives*, Cambridge Univ. Press, 1995.

a linear *ordinary* differential equation with variable coefficients. Suppose now that the temperature is specified at the two endpoints of the bar, say

$$u(0) = \alpha, \quad u(L) = \beta.$$

Our physical intuition suggests that the steady-state heat equation should have a unique solution with these boundary conditions.

a. Solve the following special case of this boundary-value problem: Find $u(x)$, defined for $0 \leq x \leq 1$ such that

$$\frac{d^2u}{dx^2} = 0, \quad u(0) = 70, \quad u(1) = 50.$$

b. Solve the following special case of this boundary-value problem: Find $u(x)$, defined for $0 \leq x \leq 1$ such that

$$\frac{d^2u}{dx^2} - u = 0, \quad u(0) = 70, \quad u(1) = 50.$$

c. Solve the following special case of this boundary-value problem: Find $u(x)$, defined for $0 \leq x \leq 1$ such that

$$\frac{d^2u}{dx^2} + x(1-x) = 0, \quad u(0) = 0, \quad u(1) = 0.$$

d. (For students with access to Mathematica) Use Mathematica to graph the solution to the following boundary-value problem: Find $u(x)$, defined for $0 \leq x \leq 1$ such that

$$\frac{d^2u}{dx^2} + (1 + x^2)u = 0, \quad u(0) = 50, \quad u(1) = 100.$$

You can do this by running the Mathematica program:

```
a = 0; b = 1; alpha = 50; beta = 100;
sol = NDSolve[ {u''[x] + (1 + x∧2) u[x] == 0,
u[a] == alpha, u[b] == beta.}, u, {x,a,b}];
Plot[ Evaluate[ u[x] /.  sol], {x,a,b}]
```

4.1.2.a. Show that the function

$$u_0(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

is a solution to the heat equation (4.6) for $t > 0$ in the case where $\kappa/(\rho\sigma) = 1$.

b. Use the chain rule with intermediate variables $\bar{x} = x - a$, $\bar{t} = t$ to show that

$$u_a(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-(x-a)^2/4t}$$

is also a solution to the heat equation.

c. Show that

$$\int_{-\infty}^{\infty} u_0(x,t)dx = 1.$$

Hint: Let I denote the integral on the left hand side and note that

$$I^2 = \frac{1}{4\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/4t}dxdy.$$

Then transform this last integral to polar coordinates.

d. Use Mathematica to sketch $u_0(x,t)$ for various values of t. What can you say about the behaviour of the function $u_0(x,t)$ as $t \to 0$?

e. By differentiating under the integral sign show that if $h : R \to R$ is any smooth function which vanishes outside a finite interval $[-L, L]$, then

$$u(x,t) = \int_{-\infty}^{\infty} u_a(x,t)h(a)da \tag{4.7}$$

is a solution to the heat equation.

REMARK: In more advanced courses it is shown that (4.7) approaches $h(x)$ as $t \to 0$. In fact, (4.7) gives a formula (for values of t which are greater than zero) for the unique solution to the heat equation on the infinite line which satisfies the initial condition $u(x,0) = h(x)$. In the next section, we will see how to solve the initial value problem for heat in a rod of finite length.

4.2 The initial value problem for the heat equation

We will now describe how to use the Fourier sine series to find the solution to an *initial value problem* for the heat equation in a rod of length L which is insulated along the sides, whose ends are kept at zero temperature. We expect that there should exist a unique function $u(x,t)$, defined for $0 \le x \le L$ and $t \ge 0$ such that

1. $u(x,t)$ satisfies the heat equation

   $$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \tag{4.8}$$

   where c is a constant.

2. $u(x,t)$ satisfies the boundary condition $u(0,t) = u(L,t) = 0$, in other words, the temperature is zero at the endpoints. (This is sometimes called the *Dirichlet boundary condition.*)

3. $u(x,t)$ satisfies the initial condition $u(x,0) = h(x)$, where $h(x)$ is a given function, the initial temperature of the rod.

In more advanced courses, it is proven that this initial value problem does in fact have a unique solution. We will shortly see how to find that solution.

Note that the heat equation itself and the boundary condition are *homogeneous* and *linear*—this means that if u_1 and u_2 satisfy these conditions, so does $c_1u_1 + c_2u_2$, for any choice of constants c_1 and c_2. We say that homogeneous linear conditions satisfy the *principal of superposition*.

Our method makes use of the dichotomy into homogeneous and nonhomogeneous conditions:

Step I. We find all of the solutions to the homogeneous linear conditions of the special form

$$u(x,t) = f(x)g(t).$$

By the superposition principle, an arbitrary linear superposition of these solutions will still be a solution.

Step II. We find the particular solution which satisfies the nonhomogeneous condition by Fourier analysis.

Let us first carry out Step I. We substitute $u(x,t) = f(x)g(t)$ into the heat equation (4.8) and obtain

$$f(x)g'(t) = c^2 f''(x)g(t).$$

Now we separate variables, putting all the functions involving t on the left, all the functions involving x on the right:

$$\frac{g'(t)}{g(t)} = c^2 \frac{f''(x)}{f(x)}.$$

The left-hand side of this equation does not depend on x, while the right-hand side does not depend on t. Hence neither side can depend upon either x or t. In other words, the two sides must equal a constant, which we denote by λ and call the *separating constant*. Our equation now becomes

$$\frac{g'(t)}{c^2 g(t)} = \frac{f''(x)}{f(x)} = \lambda,$$

which separates into two ordinary differential equations,

$$\frac{g'(t)}{c^2 g(t)} = \lambda, \quad \text{or} \quad g'(t) = \lambda c^2 g(t), \tag{4.9}$$

and

$$\frac{f''(x)}{f(x)} = \lambda, \quad \text{or} \quad f''(x) = \lambda f(x). \tag{4.10}$$

The homogeneous boundary condition $u(0,t) = u(L,t) = 0$ becomes

$$f(0)g(t) = f(L)g(t) = 0.$$

If $g(t)$ is not identically zero,

$$f(0) = f(L) = 0.$$

(If $g(t)$ is identically zero, then so is $u(x,t)$, and we obtain only the trivial solution $u \equiv 0$.)

Thus to find the nontrivial solutions to the homogeneous linear part of the problem requires us to find the nontrivial solutions to a boundary value problem for an ordinary differential equation:

$$f''(x) = \frac{d^2}{dx^2}(f(x)) = \lambda f(x), \quad f(0) = 0 = f(L). \tag{4.11}$$

We will call (4.11) the *eigenvalue problem* for the differential operator

$$\mathbf{L} = \frac{d^2}{dx^2}$$

acting on the space V_0 of well-behaved functions $f : [0,L] \to \mathbb{R}$ which vanish at the endpoints 0 and L.

In analogy with the eigenvalue problem for matrices, we can set

$$W_\lambda = \{f \in V_0 : \mathbf{L}(f) = \lambda f\},$$

for each choice of scalar λ. To solve the eigenvalue problem means to find the *eigenvalues*, those λ for which $W_\lambda \neq 0$ and for each eigenvalue the corresponding *eigenfunctions*, that is, the nonzero elements of W_λ.

We need to consider three cases. (As it turns out, only one of these will actually yield nontrivial solutions to our eigenvalue problem.)

Case 1: $\lambda = 0$. In this case, the eigenvalue problem (4.11) becomes

$$f''(x) = 0, \quad f(0) = 0 = f(L).$$

The general solution to the differential equation is $f(x) = ax + b$, and the only particular solution which satisfies the boundary condition is $f = 0$, the trivial solution.

Case 2: $\lambda > 0$. In this case, the differential equation

$$f''(x) = \lambda f(x), \quad \text{or} \quad f''(x) - \lambda f(x) = 0$$

has the general solution

$$f(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}.$$

It is convenient for us to change basis in the linear space of solutions, using

$$\cosh(\sqrt{\lambda}x) = \frac{1}{2}(e^{\sqrt{\lambda}x} + e^{-\sqrt{\lambda}x}), \qquad \sinh(\sqrt{\lambda}x) = \frac{1}{2}(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x})$$

instead of the exponentials. Then we can write

$$f(x) = a\cosh(\sqrt{\lambda}x) + b\sinh(\sqrt{\lambda}x),$$

with new constants of integration a and b. We impose the boundary conditions: first

$$f(0) = 0 \quad \Rightarrow \quad a = 0 \quad \Rightarrow \quad f(x) = b\sinh(\sqrt{\lambda}x),$$

and then

$$f(L) = 0 \quad \Rightarrow \quad b\sinh(\sqrt{\lambda}L) = 0 \quad \Rightarrow \quad b = 0 \quad \Rightarrow \quad f(x) = 0,$$

so we obtain no nontrivial solutions in this case.

Case 3: $\lambda < 0$. In this case, we set $\omega = \sqrt{-\lambda}$, and rewrite the eigenvalue problem as

$$f''(x) + \omega^2 f(x) = 0, \quad f(0) = 0 = f(L).$$

We recognize here our old friend, the differential equation of simple harmonic motion. We remember that the differential equation has the general solution

$$f(x) = a\cos(\omega x) + b\sin(\omega x).$$

Once again

$$f(0) = 0 \quad \Rightarrow \quad a = 0 \quad \Rightarrow \quad f(x) = b\sin(\omega x).$$

Now, however,

$$f(L) = 0 \quad \Rightarrow \quad b\sin(\omega L) \quad \Rightarrow \quad b = 0 \quad \text{or} \quad \sin(\omega L) = 0,$$

and hence either $b = 0$ and we obtain only the trivial solution or $\sin(\omega L) = 0$. The latter possibility will occur only if $\omega L = n\pi$, or $\omega = (n\pi/L)$, where n is an integer. In this case, we obtain

$$f(x) = b\sin(n\pi x/L).$$

Therefore, we conclude that the only nontrivial solutions to (4.11) are constant multiples of

$$f(x) = \sin\left(\frac{n\pi}{L}\right), \quad \text{with} \quad \lambda = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots. \tag{4.12}$$

This gives the solution to the eigenvalue problem for the operator $\mathbf{L} = (d^2/dx^2)$ on V_0. The eigenvalues are $\lambda = -(n\pi/L)^2$, for n a positive integer and

$$W_{-(n\pi/L)^2} \quad \text{is the one-dimensional space spanned by} \quad f(x) = \sin\left(\frac{n\pi}{L}\right).$$

For each of the solutions (4.12), we need to find a corresponding $g(t)$ solving equation (4.9),

$$g'(t) = \lambda c^2 g(t),$$

where $\lambda = -(n\pi/L)^2$. This is just the equation of exponential decay, and has the general solution

$$g(t) = be^{-(nc\pi/L)^2 t},$$

where a is a constant of integration. Thus we find that the nontrivial product solutions to the heat equation together with the homogeneous boundary condition $u(0,t) = 0 = u(L,t)$ are constant multiples of

$$u_n(x,t) = \sin(n\pi x/L)e^{-(nc\pi/L)^2 t}.$$

It follows from the principal of superposition that

$$u(x,t) = b_1 \sin(\pi x/L)e^{-(c\pi/L)^2 t} + b_2 \sin(2\pi x/L)e^{-(2c\pi/L)^2 t} + \dots \tag{4.13}$$

is a solution to the heat equation together with its homogeneous boundary conditions, for arbitrary choice of the constants $b_1, b_2, \dots$.

Step II consists of determining the constants b_n in (4.13) so that the initial condition $u(x,0) = h(x)$ is satisfied. Setting $t = 0$ in (4.13) yields

$$h(x) = u(x,0) = b_1 \sin(\pi x/L) + b_2 \sin(2\pi x/L) + \dots.$$

It follows from the theory of the Fourier sine series that h can indeed be represented as a superposition of sine functions, and we can determine the b_n's as the coefficients in the Fourier sine series of h. Using the techniques described in Section 3.3, we find that

$$b_n = \frac{2}{L}\int_0^L h(x)\sin(n\pi x/L)dx.$$

Example 1. Suppose that we want to find the function $u(x,t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, which satisfies the initial-value problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = h(x) = 4\sin x + 2\sin 2x + 7\sin 3x.$$

In this case, the nonvanishing coefficients for the Fourier sine series of h are

$$b_1 = 4, \quad b_2 = 2, \quad b_3 = 7,$$

so the solution must be

$$u(x,t) = 4(\sin x)e^{-t} + 2(\sin 2x)e^{-4t} + 7(\sin 3x)e^{-9t}.$$

Example 2. Suppose that we want to find the function $u(x,t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, which satisfies the initial-value problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = h(x),$$

where

$$h(x) = \begin{cases} x, & \text{for } 0 \le x \le \pi/2, \\ \pi - x, & \text{for } \pi/2 \le x \le \pi. \end{cases}$$

We saw in Section 3.3 that the Fourier sine series of h is

$$h(x) = \frac{4}{\pi} \sin x - \frac{4}{9\pi} \sin 3x + \frac{4}{25\pi} \sin 5x - \frac{4}{49\pi} \sin 7x + \ldots,$$

and hence

$$u(x,t) = \frac{4}{\pi}(\sin x)e^{-t} - \frac{4}{9\pi}(\sin 3x)e^{-9t} + \frac{4}{25\pi}(\sin 5x)e^{-25t} - \ldots.$$

Exercises:

4.2.1. Find the function $u(x,t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, which satisfies the following conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = \sin 2x.$$

You may assume that the nontrivial solutions to the eigenvalue problem

$$f''(x) = \lambda f(x), \quad f(0) = 0 = f(\pi)$$

are

$$\lambda = -n^2, \qquad f(x) = b_n \sin nx, \qquad \text{for } n = 1, 2, 3, \ldots,$$

where the b_n's are constants.

4.2.2. Find the function $u(x,t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, which satisfies the following conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = \sin x + 3\sin 2x - 5\sin 3x.$$

4.2.3. Find the function $u(x,t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, which satisfies the following conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = x(\pi - x).$$

(In this problem you need to find the Fourier sine series of $h(x) = x(\pi - x)$.)

4.2.4. Find the function $u(x,t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, which satisfies the following conditions:

$$\frac{1}{2t+1}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = \sin x + 3\sin 2x.$$

4.2.5. Find the function $u(x,t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, which satisfies the following conditions:

$$(t+1)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = \sin x + 3\sin 2x.$$

4.2.6.a. Find the function $w(x)$, defined for $0 \le x \le \pi$, such that

$$\frac{d^2 w}{dx^2} = 0, \qquad w(0) = 10, \qquad w(\pi) = 50.$$

b. Find the general solution to the following boundary value problem for the heat equation: Find the functions $u(x,t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, such that

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = 10, \qquad u(\pi,t) = 50. \tag{4.14}$$

(Hint: Let $v = u - w$, where w is the solution to part a. Determine what conditions v must satisfy.)

c. Find the particular solution to (4.14) which in addition satisfies the initial condition

$$u(x,0) = 10 + \frac{40}{\pi}x + 2\sin x - 5\sin 2x.$$

4.2.7. The method described in this section can also be used to solve an initial value problem for the heat equation in which the Dirichlet boundary condition $u(0,t) = u(L,t) = 0$ is replaced by the *Neumann boundary condition*

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0. \tag{4.15}$$

(Physically, this corresponds to insulated endpoints, from which no heat can enter or escape.) In this case separation of variables leads to a slightly different eigenvalue problem, which consists of finding the nontrivial solutions to

$$f''(x) = \frac{d^2}{dx^2}(f(x)) = \lambda f(x), \quad f'(0) = 0 = f'(L).$$

a. Solve this eigenvalue problem. (Hint: The solution should involve cosines instead of sines.)

b. Find the general solution to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

subject to the Neumann boundary condition (4.15).

c. Find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, such that:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0, \quad u(x,0) = 3\cos x + 7\cos 2x.$$

4.2.8. We can also treat a mixture of Dirichlet and Neumann conditions, say

$$u(0,t) = \frac{\partial u}{\partial x}(L,t) = 0. \tag{4.16}$$

In this case separation of variables leads to the eigenvalue problem which consists of finding the nontrivial solutions to

$$f''(x) = \frac{d^2}{dx^2}(f(x)) = \lambda f(x), \quad f(0) = 0 = f'(L).$$

a. Solve this eigenvalue problem.

b. Find the general solution to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

subject to the mixed boundary condition (4.16).

c. Find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, such that:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0, \quad u(x,0) = 4\sin(x/2) + 12\sin(3x/2).$$

4.2.9.a. Find the eigenvalues and corresponding eigenfunctions for the differential operator

$$\mathbf{L} = \frac{d^2}{dt^2} - 3,$$

which acts on the space V_0 of well-behaved functions $f : [0,\pi] \to \mathbb{R}$ which vanish at the endpoints 0 and π by

$$\mathbf{L}(f) = \frac{d^2 f}{dt^2} - 3f.$$

b. Find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, which satisfies the following conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 3u, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = \sin x + 3\sin 2x.$$

4.3 Numerical solutions to the heat equation

There is another method which is sometimes used to treat the initial value problem described in the preceding section, a numerical method based upon "finite differences." Although it yields only approximate solutions, it can be applied in some cases with variable coefficients when it would be impossible to apply Fourier analysis in terms of sines and cosines. However, for simplicity, we will describe only the case where ρ and k are constant, and in fact we will assume that $c^2 = L = 1$.

Thus we seek the function $u(x,t)$, defined for $0 \le x \le 1$ and $t \ge 0$, which solves the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

subject to the boundary conditions $u(0,t) = u(1,t) = 0$ and the initial condition $u(x,0) = h(x)$, where $h(x)$ is a given function, representing the initial temperature.

For any fixed choice of t_0 the function $u(x,t_0)$ is an element of V_0, the space of piecewise smooth functions defined for $0 \le x \le 1$ which vanish at the endpoints. Our idea is to replace the "infinite-dimensional" space V_0 by a finite-dimensional Euclidean space $\mathbb{R}^{n-1}$ and reduce the partial differential equation to a system of ordinary differential equations. This corresponds to utilizing a discrete model for heat flow rather than a continuous one.

For $0 \le i \le n$, let $x_i = i/n$ and

$$u_i(t) = u(x_i, t) = \text{the temperature at } x_i \text{ at time } t.$$

Since $u_0(t) = 0 = u_n(t)$ by the boundary conditions, the temperature at time t is specified by

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ u_{n-1}(t) \end{pmatrix},$$

a vector-valued function of one variable. The initial condition becomes

$$\mathbf{u}(0) = \mathbf{h}, \qquad \text{where} \qquad \mathbf{h} = \begin{pmatrix} h(x_1) \\ h(x_2) \\ \cdot \\ h(x_{n-1}) \end{pmatrix}.$$

We can approximate the first-order partial derivative by a difference quotient:

$$\frac{\partial u}{\partial x}\left(\frac{x_i + x_{i+1}}{2}, t\right) \doteq \frac{u_{i+1}(t) - u_i(t)}{x_{i+1} - x_i} = \frac{[u_{i+1}(t) - u_i(t)]}{1/n} = n[u_{i+1}(t) - u_i(t)].$$

Similarly, we can approximate the second-order partial derivative:

$$\frac{\partial^2 u}{\partial x^2}(x_i, t) \doteq \frac{\frac{\partial u}{\partial x}\left(\frac{x_i + x_{i+1}}{2}, t\right) - \frac{\partial u}{\partial x}\left(\frac{x_{i-1} + x_i}{2}, t\right)}{1/n}$$

$$\doteq n \left[\frac{\partial u}{\partial x}\left(\frac{x_i + x_{i+1}}{2}, t\right) - \frac{\partial u}{\partial x}\left(\frac{x_{i-1} + x_i}{2}, t\right) \right]$$

$$\doteq n^2[u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)].$$

Thus the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

can be approximated by a system of ordinary differential equations

$$\frac{du_i}{dt} = n^2(u_{i-1} - 2u_i + u_{i+1}).$$

This is a first order linear system which can be presented in vector form as

$$\frac{d\mathbf{u}}{dt} = n^2 P\mathbf{u}, \quad \text{where} \quad P = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & \cdots & -2 \end{pmatrix},$$

the last matrix having $n-1$ rows and $n-1$ columns. Finally, we can rewrite this as

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}, \qquad \text{where} \qquad A = n^2 P, \tag{4.17}$$

a first order linear homogenous system of ordinary differential equations with constant coefficients. In the limit as $n \to \infty$ one could use Mathematica software to check that the eigenvalues of A approach the eigenvalues of d^2/dx^2 as determined in the preceding section, and the eigenvectors approximate more and more closely the standard orthonormal basis of sine functions.

One could continue constructing a numerical method for solution of our initial value problem by means of another discretization, this time in the time direction. We could do this via the familiar Cauchy-Euler method for finding numerical solutions to the linear system (4.17). This method for finding approximate solutions to the heat equation is often called the *method of finite differences*. With sufficient effort, one could construct a computer program, using Mathematica or some other software package, to implement it.

More advanced courses on numerical analysis often treat the finite difference method in detail.[3] For us, however, the main point of the method of finite differences is that it provides considerable insight into the theory behind the heat equation. It shows that the heat equation can be thought of as arising from a system of ordinary differential equations when the number of dependent variables goes to infinity. It is sometimes the case that either a partial differential equation or a system of ordinary differential equations with a large or even

[3]For further discussion of this method one can refer to numerical analysis books, such as Burden and Faires, *Numerical analysis*, Seventh edition, Brooks Cole Publishing Company, 2000.

infinite number of unknowns can give an effective model for the same physical phenomenon. This partially explains, for example, why quantum mechanics possesses two superficially different formulations, via Schrödinger's partial differential equation or via "infinite matrices" in Heisenberg's "matrix mechanics."

4.4 The vibrating string

Our next goal is to derive the equation which governs the motion of a vibrating string. We consider a string of length L stretched out along the x-axis, one end of the string being at $x = 0$ and the other being at $x = L$. We assume that the string is free to move only in the vertical direction. Let

$$u(x,t) = \text{vertical displacement of the string at the point } x \text{ at time } t.$$

We will derive a partial differential equation for $u(x,t)$. Note that since the ends of the string are fixed, we must have $u(0,t) = 0 = u(L,t)$ for all t.

It will be convenient to use the "configuration space" V_0 described in Section 3.3. An element $u(x) \in V_0$ represents a configuration of the string at some instant of time. We will assume that the *potential energy* in the string when it is in the configuration $u(x)$ is

$$V(u(x)) = \int_0^L \frac{T}{2}\left(\frac{du}{dx}\right)^2 dx, \tag{4.18}$$

where T is a constant, called the *tension* of the string.

Indeed, we could imagine that we have devised an experiment that measures the potential energy in the string in various configurations, and has determined that (4.18) does indeed represent the total potential energy in the string. On the other hand, this expression for potential energy is quite plausible for the following reason: We could imagine first that the amount of energy in the string should be proportional to the amount of stretching of the string, or in other words, proportional to the length of the string. From vector calculus, we know that the length of the curve $u = u(x)$ is given by the formula

$$\text{Length} = \int_0^L \sqrt{1 + (du/dx)^2}\, dx.$$

But when du/dx is small,

$$\left[1 + \frac{1}{2}\left(\frac{du}{dx}\right)^2\right]^2 = 1 + \left(\frac{du}{dx}\right)^2 + \text{a small error},$$

and hence

$$\sqrt{1 + (du/dx)^2} \quad \text{is closely approximated by} \quad 1 + \frac{1}{2}(du/dx)^2.$$

Thus to a first order of approximation, the amount of energy in the string should be proportional to

$$\int_0^L \left[1 + \frac{1}{2}\left(\frac{du}{dx}\right)^2\right] dx = \int_0^L \frac{1}{2}\left(\frac{du}{dx}\right)^2 dx + \text{constant}.$$

Letting T denote the constant of proportionality yields

$$\text{energy in string} = \int_0^L \frac{T}{2}\left(\frac{du}{dx}\right)^2 dx + \text{constant}.$$

Potential energy is only defined up to addition of a constant, so we can drop the constant term to obtain (4.18).

The force acting on a portion of the string when it is in the configuration $u(x)$ is determined by an element $F(x)$ of V_0. We imagine that the force acting on the portion of the string from x to $x + dx$ is $F(x)dx$. When the force pushes the string through an infinitesimal displacement $\xi(x) \in V_0$, the total *work* performed by $F(x)$ is then the "sum" of the forces acting on the tiny pieces of the string, in other words, the work is the "inner product" of F and ξ,

$$\langle F(x), \xi(x)\rangle = \int_0^L F(x)\xi(x)dx.$$

(Note that the inner product we use here differs from the one used in Section 3.3 by a constant factor.)

On the other hand this work is the amount of potential energy lost when the string undergoes the displacement:

$$\begin{aligned}\langle F(x), \xi(x)\rangle &= \int_0^L \frac{T}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx - \int_0^L \frac{T}{2}\left(\frac{\partial(u+\xi)}{\partial x}\right)^2 dx \\ &= -T\int_0^L \frac{\partial u}{\partial x}\frac{\partial \xi}{\partial x}dx + \int_0^L \frac{T}{2}\left(\frac{\partial \xi}{\partial x}\right)^2 dx.\end{aligned}$$

We are imagining that the displacement ξ is infinitesimally small, so terms containing the square of ξ or the square of a derivative of ξ can be ignored, and hence

$$\langle F(x), \xi(x)\rangle = -T\int_0^L \frac{\partial u}{\partial x}\frac{\partial \xi}{\partial x}dx.$$

Integration by parts yields

$$\langle F(x), \xi(x)\rangle = T\int_0^L \frac{\partial^2 u}{\partial x^2}\xi(x)dx - T\left(\frac{\partial u}{\partial x}\xi\right)(L) - T\left(\frac{\partial u}{\partial x}\xi\right)(0).$$

Since $\xi(0) = \xi(L) = 0$, we conclude that

$$\int_0^L F(x)\xi(x)dx = \langle F(x), \xi(x)\rangle = T\int_0^L \frac{\partial^2 u}{\partial x^2}\xi(x)dx.$$

Since this formula holds for *all* infinitesimal displacements $\xi(x)$, we must have

$$F(x) = T\frac{\partial^2 u}{\partial x^2},$$

for the force density per unit length.

Now we apply Newton's second law, force = mass × acceleration, to the function $u(x,t)$. The force acting on a tiny piece of the string of length dx is $F(x)dx$, while the mass of this piece of string is just ρdx, where ρ is the density of the string. Thus Newton's law becomes

$$T\frac{\partial^2 u}{\partial x^2}dx = \rho dx \frac{\partial^2 u}{\partial t^2}.$$

If we divide by ρdx, we obtain the *wave equation*,

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}, \qquad \text{or} \qquad \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2},$$

where $c^2 = T/\rho$.

Just as in the preceding section, we could approximate this partial differential equation by a system of ordinary differential equations. Assume that the string has length $L = 1$ and set $x_i = i/n$ and

$$u_i(t) = u(x_i, t) = \text{the displacement of the string at } x_i \text{ at time } t.$$

Then the function $u(x,t)$ can be approximated by the vector-valued function

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ u_{n-1}(t) \end{pmatrix}$$

of one variable, just as before. The wave equation is then approximated by the system of ordinary differential equations

$$\frac{d^2\mathbf{u}}{dt^2} = c^2 n^2 P\mathbf{u},$$

where P is the $(n-1)\times(n-1)$ matrix described in the preceding section. Thus the differential operator

$$\mathbf{L} = \frac{d^2}{dx^2} \quad \text{is approximated by the symmetric matrix} \quad n^2 P,$$

and we expect solutions to the wave equation to behave like solutions to a mechanical system of weights and springs with a large number of degrees of freedom.

Exercises:

4.4.1.a. Show that if $f : \mathbb{R} \to \mathbb{R}$ is any well-behaved function of one variable,

$$u(x,t) = f(x+ct)$$

is a solution to the partial differential equation

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0.$$

(Hint: Use the "chain rule.")

b. Show that if $g : \mathbb{R} \to \mathbb{R}$ is any well-behaved function of one variable,

$$u(x,t) = g(x-ct)$$

is a solution to the partial differential equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0.$$

c. Show that for any choice of well-behaved functions f and g, the function

$$u(x,t) = f(x+ct) + g(x-ct)$$

is a solution to the differential equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = \left[\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right]\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right) = 0.$$

Remark: This gives a very explicit general solution to the equation for the vibrations of an infinitely long string.

d. Show that

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2}$$

is a solution to the initial value problem

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0, \qquad u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

4.4.2. Show that if the tension and density of a string are given by the constant T and the variable function $\rho(x)$ respectively, then the motion of the string is governed by the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho(x)}\frac{\partial^2 u}{\partial x^2}.$$

4.5 The initial value problem for the vibrating string

The Fourier sine series can also be used to find the solution to an *initial value problem* for the vibrating string with fixed endpoints at $x = 0$ and $x = L$. We formulate this problem as follows: we seek a function $u(x,t)$, defined for $0 \leq x \leq L$ and $t \geq 0$ such that

1. $u(x,t)$ satisfies the wave equation
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{4.19}$$
where c is a constant.

2. $u(x,t)$ satisfies the boundary condition $u(0,t) = u(L,t) = 0$, i.e. the displacement of the string is zero at the endpoints.

3. $u(x,t)$ satisfies the initial conditions
$$u(x,0) = h_1(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = h_2(x),$$
where $h_1(x)$ and $h_2(x)$ are given functions, the initial position and velocity of the string.

Note that the wave equation itself and the boundary condition are *homogeneous* and *linear*, and therefore satisfy the principal of superposition.

Once again, we find the solution to our problem in two steps:

Step I. We find all of the solutions to the homogeneous linear conditions of the special form
$$u(x,t) = f(x)g(t).$$

Step II. We find the superposition of these solution which satisfies the nonhomogeneous initial conditions by means of Fourier analysis.

To carry out Step I, we substitute $u(x,t) = f(x)g(t)$ into the wave equation (4.19) and obtain
$$f(x)g''(t) = c^2 f''(x)g(t).$$
We separate variables, putting all the functions involving t on the left, all the functions involving x on the right:
$$\frac{g''(t)}{g(t)} = c^2 \frac{f''(x)}{f(x)}.$$
Once again, the left-hand side of this equation does not depend on x, while the right-hand side does not depend on t, so neither side can depend upon either x or t. Therefore the two sides must equal a constant λ, and our equation becomes
$$\frac{g''(t)}{c^2 g(t)} = \frac{f''(x)}{f(x)} = \lambda,$$

which separates into two ordinary differential equations,

$$\frac{g''(t)}{c^2 g(t)} = \lambda, \quad \text{or} \quad g''(t) = \lambda c^2 g(t), \tag{4.20}$$

and

$$\frac{f''(x)}{f(x)} = \lambda, \quad \text{or} \quad f''(x) = \lambda f(x). \tag{4.21}$$

Just as in the case of the heat equation, the homogeneous boundary condition $u(0,t) = u(L,t) = 0$ becomes

$$f(0)g(t) = f(L)g(t) = 0,$$

and assuming that $g(t)$ is not identically zero, we obtain

$$f(0) = f(L) = 0.$$

Thus once again we need to find the nontrivial solutions to the boundary value problem,

$$f''(x) = \frac{d^2}{dx^2}(f(x)) = \lambda f(x), f(0) = 0 = f(L),$$

and just as before, we find that the the only nontrivial solutions are constant multiples of

$$f(x) = \sin(n\pi x/L), \quad \text{with} \quad \lambda = -(n\pi/L)^2, \quad n = 1, 2, 3, \ldots.$$

For each of these solutions, we need to find a corresponding $g(t)$ solving equation (4.20),

$$g''(t) = -(n\pi/L)^2 c^2 g(t), \quad \text{or} \quad g''(t) + (n\pi/L)^2 c^2 g(t) = 0.$$

This is just the equation of simple harmonic motion, and has the general solution

$$g(t) = a\cos(nc\pi t/L) + b\sin(nc\pi t/L),$$

where a and b are constants of integration. Thus we find that the nontrivial product solutions to the wave equation together with the homogeneous boundary condition $u(0,t) = 0 = u(L,t)$ are constant multiples of

$$u_n(x,t) = [a_n\cos(nc\pi t/L) + b_n\sin(nc\pi t/L)]\sin(n\pi x/L).$$

The general solution to the wave equation together with this boundary condition is an arbitrary superposition of these product solutions:

$$\begin{aligned} u(x,t) = {} & [a_1\cos(c\pi t/L) + b_1\sin(c\pi t/L)]\sin(\pi x/L) \\ & + [a_2\cos(2c\pi t/L) + b_2\sin(2c\pi t/L)]\sin(2\pi x/L) + \ldots. \end{aligned} \tag{4.22}$$

The vibration of the string is a superposition of a *fundamental mode* which has frequency

$$\frac{c\pi}{L}\frac{1}{2\pi} = \frac{c}{2L} = \frac{\sqrt{T/\rho}}{2L},$$

and higher modes which have frequencies which are exact integer multiples of this frequency.

Step II consists of determining the constants a_n and b_n in (4.22) so that the initial conditions

$$u(x,0) = h_1(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = h_2(x)$$

are satisfied. Setting $t = 0$ in (4.22) yields

$$h_1(x) = u(x,0) = a_1 \sin(\pi x/L) + a_2 \sin(2\pi x/L) + \ldots,$$

so we see that the a_n's are the coefficients in the Fourier sine series of h_1.

If we differentiate equation(4.22) with respect to t, we find that

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) = [&\frac{-c\pi}{L} a_1 \sin(c\pi t/L) + \frac{c\pi}{L} b_1 \cos(c\pi t/L)] \sin(\pi x/L) \\ &+ \frac{-2c\pi}{L}[a_2 \sin(2c\pi t/L) + \frac{c\pi}{L} b_2 \cos(2c\pi t/L)] \sin(2\pi x/L) + \ldots,\end{aligned}$$

and setting $t = 0$ yields

$$h_2(x) = \frac{\partial u}{\partial t}(x,0) = \frac{c\pi}{L} b_1 \sin(\pi x/L) + \frac{2c\pi}{L} b_2 \sin(2\pi x/L) + \ldots.$$

We conclude that

$$\frac{nc\pi}{L} b_n = \text{the } n\text{-th coefficient in the Fourier sine series of } h_2(x). \tag{4.23}$$

Example 1. Suppose that we want to find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, which satisfies the initial-value problem:

$$\begin{aligned}&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0, \\ &\qquad\qquad u(x,0) = 5\sin x + 12 \sin 2x + 6 \sin 3x, \quad \frac{\partial u}{\partial t}(x,0) = 0.\end{aligned}$$

In this case, $h_2 = 0$ and the first three coefficients for the Fourier sine series of h_1 are

$$a_1 = 5, \quad a_2 = 12, \quad a_3 = 6,$$

all the other coefficients being zero. So the solution in this case must be

$$u(x,t) = 5 \sin x \cos t + 12 \sin 2x \cos 2t + 6 \sin 3x \cos 3t.$$

Example 2. Suppose that we want to find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, which satisfies the initial-value problem:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0,$$
$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 3\sin x + 8\sin 2x + 15\sin 3x.$$

Note that $c = 1$ and $L = \pi$. In this example, $h_1 = 0$, and it follows from (4.23) that the first three coefficients for the Fourier sine series of h_2 are

$$b_1 = 3, \quad 2b_2 = 8, \quad 3b_3 = 6,$$

all the other coefficients being zero. So the solution in this case must be

$$u(x,t) = 3\sin x \sin t + 4\sin 2x \sin 2t + 2\sin 3x \sin 3t.$$

Exercises:

4.5.1 What happens to the frequency of the fundamental mode of oscillation of a vibrating string when the length of the string is doubled? When the tension on the string is doubled? When the density of the string is doubled?

4.5.2. Find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, which satisfies the following conditions:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0,$$
$$u(x,0) = \sin 2x, \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

You may assume that the nontrivial solutions to the eigenvalue problem

$$f''(x) = \lambda f(x), \quad f(0) = 0 = f(\pi)$$

are

$$\lambda = -n^2, \qquad f(x) = b_n \sin nx, \qquad \text{for } n = 1, 2, 3, \ldots,$$

where the b_n's are constants.

4.5.3. Find the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, which satisfies the following conditions:

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0,$$
$$u(x,0) = \sin x + 3\sin 2x - 5\sin 3x, \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

4.5.4. Find the function $u(x,t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, which satisfies the following conditions:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0,$$

$$u(x,0) = x(\pi - x), \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

4.5.5. Find the function $u(x,t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, which satisfies the following conditions:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0,$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = \sin x + \sin 2x.$$

4.5.6. Find the function $u(x,t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, which satisfies the following conditions:

$$\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0,$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 7\sin x + 3\sin 2x + \sin 3x.$$

4.5.7. Find the function $u(x,t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, which satisfies the following conditions:

$$(t+1)^2\frac{\partial^2 u}{\partial t^2} + (t+1)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0,$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \frac{1}{n!} \sin nt.$$

4.5.8. (For students with access to Mathematica) a. Find the first ten coefficients of the Fourier sine series for

$$h(x) = x - x^4$$

by running the following Mathematica program

```
f[n_] := 2 NIntegrate[(x - x^4) Sin[n Pi x], {x,0,1}];
b = Table[f[n], {n,1,10}]
```

b. Find the first ten terms of the solution to the initial value problem for a vibrating string,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0,$$

$$u(x,0) = x - x^4, \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

c. Construct a sequence of sketches of the positions of the vibrating string at the times $t_i = ih$, where $h = .1$ by running the Mathematica program:

```
vibstring = Table[
    Plot[
        Sum[ b[n] Sin[n Pi x] Cos[n Pi t], {n,1,10}],
        {x,0,1}, PlotRange -> {-1,1}
    ], {t,0,1.,.1}
]
```

d. Select the cell containing `vibstring` and animate the sequence of graphics by running "Animate selected graphics," from the Cell menu.

4.6 Heat flow in a circular wire

The theory of Fourier series can also be used to solve the *initial value problem* for the heat equation in a circular wire of radius 1 which is insulated along the sides. In this case, we seek a function $u(\theta, t)$, defined for $\theta \in \mathbb{R}$ and $t \geq 0$ such that

1. $u(\theta, t)$ satisfies the heat equation
$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial \theta^2}, \tag{4.24}$$
where c is a constant.

2. $u(\theta, t)$ is periodic of period 2π in the variable θ; in other words,
$$u(\theta + 2\pi, t) = u(\theta, t), \qquad \text{for all } \theta \text{ and } t.$$

3. $u(\theta, t)$ satisfies the initial condition $u(\theta, 0) = h(\theta)$, where $h(\theta)$ is a given function, periodic of period 2π, the initial temperature of the wire.

Once again the heat equation itself and the periodicity condition are homogeneous and linear, so they must be dealt with first. Once we have found the general solution to the homogeneous conditions

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial \theta^2}, \quad u(\theta + 2\pi, t) = u(\theta, t),$$

we will be able to find the particular solution which satisfies the initial condition

$$u(\theta, 0) = h(\theta)$$

by the theory of Fourier series.

Thus we substitute $u(\theta, t) = f(\theta)g(t)$ into the heat equation (4.24) to obtain

$$f(\theta)g'(t) = c^2 f''(\theta)g(t)$$

and separate variables:

$$\frac{g'(t)}{g(t)} = c^2 \frac{f''(\theta)}{f(\theta)}.$$

The left-hand side of this equation does not depend on θ, while the right-hand side does not depend on t, so neither side can depend upon either θ or t, and we can write

$$\frac{g'(t)}{c^2 g(t)} = \frac{f''(\theta)}{f(\theta)} = \lambda,$$

where λ is a constant. We thus obtain two ordinary differential equations,

$$\frac{g'(t)}{c^2 g(t)} = \lambda, \quad \text{or} \quad g'(t) = \lambda c^2 g(t), \tag{4.25}$$

and

$$\frac{f''(\theta)}{f(\theta)} = \lambda, \quad \text{or} \quad f''(\theta) = \lambda f(\theta). \tag{4.26}$$

The periodicity condition $u(\theta + \pi, t) = u(\theta, t)$ becomes

$$f(\theta + 2\pi)g(t) = f(\theta)g(t),$$

and if $g(t)$ is not identically zero, we must have

$$f(\theta + 2\pi) = f(\theta).$$

Thus to find the nontrivial solutions to the homogeneous linear part of the problem requires us to find the nontrivial solutions to:

$$f''(\theta) = \frac{d^2}{d\theta^2}(f(\theta)) = \lambda f(\theta), \quad f(\theta + 2\pi) = f(\theta). \tag{4.27}$$

We will call (4.11) the *eigenvalue problem* for the differential operator

$$\mathbf{L} = \frac{d^2}{d\theta^2}$$

acting on the space V of smooth functions which are periodic of period 2π.

As before, in analogy with the eigenvalue problem for matrices, we can set

$$W_\lambda = \{f \in V : \mathbf{L}(f) = \lambda f\},$$

for each choice of scalar λ, and our goal is to find the eigenvalues, the values of λ for which $W_\lambda \neq 0$ and the corresponding eigenfunctions, the nonzero elements of W_λ. We need to consider three cases.

Case 1: $\lambda = 0$. In this case, the eigenvalue problem (4.27) becomes

$$f''(\theta) = 0, \quad f(\theta + 2\pi) = f(\theta).$$

The general solution to the differential equation is $f(\theta) = a + b\theta$, and

$$a + b(\theta + 2\pi) = a + b(\theta) \Rightarrow b = 0.$$

Thus the only solution in this case is that where f is constant, and to be consistent with our Fourier series conventions, we write $f(\theta) = a_0/2$, where a_0 is a constant.

Case 2: $\lambda > 0$. In this case, the differential equation

$$f''(\theta) = \lambda f(\theta), \quad \text{or} \quad f''(\theta) - \lambda f(\theta) = 0$$

has the general solution

$$f(\theta) = ae^{(\sqrt{\lambda}\theta)} + be^{-(\sqrt{\lambda}\theta)}.$$

Note that

$$a \neq 0 \Rightarrow f(\theta) \to \pm\infty \quad \text{as} \quad \theta \to \infty,$$

while

$$b \neq 0 \Rightarrow f(\theta) \to \pm\infty \quad \text{as} \quad \theta \to -\infty.$$

Neither of these is consistent with the periodicity conditions $f(\theta + 2\pi) = f(\theta)$, because periodic functions are bounded, so we conclude that $a = b = 0$, and we obtain no nontrivial solutions in this case.

Case 3: $\lambda < 0$. In this case, we set $\omega = \sqrt{-\lambda}$, and rewrite the eigenvalue problem as

$$f''(\theta) + \omega^2 f(\theta) = 0, \quad f(\theta + 2\pi) = f(\theta).$$

We recognize once again our old friend, the differential equation of simple harmonic motion, which has the general solution

$$f(\theta) = a\cos(\omega\theta) + b\sin(\omega\theta) = A\sin(\omega(\theta - \theta_0)).$$

The periodicity condition $f(\theta + 2\pi) = f(\theta)$ implies that $\omega = n$, where n is an integer, which we can assume is positive, and

$$f(\theta) = a_n\cos(n\theta) + b_n\sin(n\theta).$$

Therefore, we conclude that the nontrivial solutions to (4.27) are

$$\lambda = 0 \quad \text{and} \quad f(\theta) = \frac{a_0}{2},$$

and

$$\lambda = -n^2 \quad \text{where } n \text{ is a positive integer and} \quad f(\theta) = a_n\cos(n\theta) + b_n\sin(n\theta).$$

This gives the solution to the eigenvalue problem for the operator $\mathbf{L} = (d^2/d\theta^2)$ on V. The eigenvalues are $\lambda = -n^2$, for n a nonnegative integer. Moreover,

$$W_0 \quad \text{is the one-dimensional space spanned by} \quad f(\theta) = 1,$$

while

$$W_{-n^2} \quad \text{is the two-dimensional space spanned by} \quad \cos(n\theta) \quad \text{and} \quad \sin(n\theta).$$

In general, we say that an eigenvalue λ has *multiplicity* k when the corresponding eigenspace W_λ has dimension k; thus for the operator $\mathbf{L} = (d^2/d\theta^2)$ on V, all the nonzero eigenvalues have multiplicity two.

Next we need to find the corresponding solutions to (4.25)

$$g'(t) = \lambda c^2 g(t),$$

for $\lambda = 0, -1, -4, -9, \ldots, -n^2, \ldots$. As before, we find that the solutions are

$$g(t) = (\text{constant})e^{-n^2c^2t},$$

where c is a constant. Thus the product solutions to the homogeneous part of the problem are

$$u_0(\theta, t) = \frac{a_0}{2}, \quad u_n(\theta, t) = [a_n \cos(n\theta) + b_n \sin(n\theta)]e^{-n^2c^2t},$$

where $n = 1, 2, 3, \ldots$.

Now we apply the superposition principle—an arbitrary superposition of these product solutions must again be a solution. Thus

$$u(\theta, t) = \frac{a_0}{2} + \Sigma_{n=1}^{\infty}[a_n \cos(n\theta) + b_n \sin(n\theta)]e^{-n^2c^2t} \tag{4.28}$$

is a periodic solution of period 2π to the heat equation (4.24).

To finish the solution to our problem, we must impose the initial condition

$$u(\theta, 0) = h(\theta).$$

But setting $t = 0$ in (4.28) yields

$$\frac{a_0}{2} + \Sigma_{n=1}^{\infty}[a_n \cos(n\theta) + b_n \sin(n\theta)] = h(\theta),$$

so the constants $a_0, a_1, \ldots, b_1, \ldots$ are just the Fourier coefficients of $h(\theta)$. Thus the solution to our initial value problem is just (4.28) in which the constants a_k and b_k can be determined via the familiar formulae

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos k\theta d\theta, \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\sin k\theta d\theta.$$

Note that as $t \to \infty$ the temperature in the circular wire approaches the constant value $a_0/2$.

Exercises:

4.6.1. Find the function $u(\theta, t)$, defined for $t \geq 0$ and arbitrary θ, which satisfies the following conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \theta^2}, \quad u(\theta + 2\pi, t) = u(\theta, t), \quad u(\theta, 0) = 2 + \sin\theta - \cos 3\theta.$$

You may assume that the nontrivial solutions to the eigenvalue problem

$$f''(\theta) = \lambda f(\theta), \quad f(\theta + 2\pi) = f(\theta)$$

are

$$\lambda = 0 \quad \text{and} \quad f(\theta) = \frac{a_0}{2},$$

and

$$\lambda = -n^2 \quad \text{and} \quad f(\theta) = a_n \cos n\theta + b_n \sin n\theta, \qquad \text{for } n = 1, 2, 3, \ldots,$$

where the a_n's and b_n's are constants.

4.6.2. Find the function $u(\theta, t)$, defined for $t \geq 0$ and arbitrary θ, which satisfies the following conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \theta^2}, \quad u(\theta + 2\pi, t) = u(\theta, t),$$
$$u(\theta, 0) = |\theta|, \qquad \text{for } \theta \in [-\pi, \pi].$$

4.6.3. Find the function $u(\theta, t)$, defined for $t \geq 0$ and arbitrary θ, which satisfies the following conditions:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial \theta^2}, \quad u(\theta + 2\pi, t) = u(\theta, t),$$
$$u(\theta, 0) = 2 + \sin\theta - \cos 3\theta, \quad \frac{\partial u}{\partial t}(\theta, 0) = 0.$$

4.6.4. Find the function $u(\theta, t)$, defined for $t \geq 0$ and arbitrary θ, which satisfies the following conditions:

$$\frac{1}{2t+3}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \theta^2}, \quad u(\theta + 2\pi, t) = u(\theta, t), \quad u(\theta, 0) = 3 + \sum_{n=1}^{\infty} \frac{1}{n!}\sin(n\theta).$$

4.7 Sturm-Liouville Theory*

We would like to be able to analyze heat flow in a bar even if the specific heat $\sigma(x)$, the density $\rho(x)$ and the thermal conductivity $\kappa(x)$ vary from point to point. As we saw in Section 4.1, this leads to consideration of the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{\rho(x)\sigma(x)}\frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u}{\partial x}\right), \tag{4.29}$$

where the functions $\rho(x)$, $\sigma(x)$ and $\kappa(x)$ are positive.

We imagine that the bar is situated along the x-axis with its endpoints situated at $x = a$ and $x = b$. As in the constant coefficient case, we expect that there should exist a unique function $u(x,t)$, defined for $a \leq x \leq b$ and $t \geq 0$ such that

1. $u(x,t)$ satisfies the heat equation (4.29).
2. $u(x,t)$ satisfies the boundary condition $u(a,t) = u(b,t) = 0$.
3. $u(x,t)$ satisfies the initial condition $u(x,0) = h(x)$, where $h(x)$ is a given function, defined for $x \in [a,b]$, the initial temperature of the bar.

Just as before, we substitute $u(x,t) = f(x)g(t)$ into (**??**) and obtain

$$f(x)g'(t) = \frac{1}{\rho(x)\sigma(x)}\frac{d}{dx}\left(\kappa(x)\frac{df}{dx}(x)\right)g(t).$$

Once again, we separate variables, putting all the functions involving t on the left, all the functions involving x on the right:

$$\frac{g'(t)}{g(t)} = \frac{1}{\rho(x)\sigma(x)}\frac{d}{dx}\left(\kappa(x)\frac{df}{dx}(x)\right)\frac{1}{f(x)}.$$

As usual, the two sides must equal a constant, which we denote by λ, and our equation separates into two ordinary differential equations,

$$g'(t) = \lambda g(t), \tag{4.30}$$

and

$$\frac{1}{\rho(x)\sigma(x)}\frac{d}{dx}\left(\kappa(x)\frac{df}{dx}(x)\right) = \lambda f(x). \tag{4.31}$$

Under the assumption that u is not identically zero, the boundary condition $u(a,t) = u(b,t) = 0$ yields

$$f(a) = f(b) = 0.$$

Thus to find the nontrivial solutions to the homogeneous linear part of the problem, we need to find the nontrivial solutions to the boundary value problem:

$$\frac{1}{\rho(x)\sigma(x)}\frac{d}{dx}\left(\kappa(x)\frac{df}{dx}(x)\right) = \lambda f(x), \qquad f(a) = 0 = f(b). \tag{4.32}$$

We call this the *eigenvalue problem* or *Sturm-Liouville problem* for the differential operator

$$\mathbf{L} = \frac{1}{\rho(x)\sigma(x)}\frac{d}{dx}\left(\kappa(x)\frac{d}{dx}\right),$$

which acts on the space V_0 of well-behaved functions $f : [a,b] \to \mathbb{R}$ which vanish at the endpoints a and b. The *eigenvalues* of $\mathbf{L}$ are the constants λ for which (4.32) has nontrivial solutions. Given an eigenvalue λ, the corresponding *eigenspace* is

$$W_\lambda = \{f \in V_0 : \ f \text{ satisfies } (4.32)\}.$$

Nonzero elements of the eigenspaces are called *eigenfunctions*.

If the functions $\rho(x)$, $\sigma(x)$ and $\kappa(x)$ are complicated, it may be impossible to solve this eigenvalue problem explicitly, and one may need to employ numerical methods to obtain approximate solutions. Nevertheless, it is reassuring to know that the theory is quite parallel to the constant coefficient case that we treated in previous sections. The following theorem, due to the nineteenth century mathematicians Sturm and Liouville, is proven in more advanced texts:[4]

Theorem. *Suppose that $\rho(x)$, $\sigma(x)$ and $\kappa(x)$ are smooth functions which are positive on the interval $[a,b]$. Then all of the eigenvalues of $\mathbf{L}$ are negative real numbers, and each eigenspace is one-dimensional. Moreover, the eigenvalues can be arranged in a sequence*

$$0 > \lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots,$$

with $\lambda_n \to -\infty$. Finally, every well-behaved function can be represented on $[a,b]$ as a convergent sum of eigenfunctions.

Suppose that $f_1(x)$, $f_2(x)$, $\ldots$, $f_n(x)$, $\ldots$ are eigenfunctions corresponding to the eigenvalues λ_1, λ_2, $\ldots$, λ_n, $\ldots$. Then the general solution to the heat equation (**??**) together with the boundary conditions $u(a,t) = u(b,t) = 0$ is

$$u(x,t) = \sum_{n=0}^{\infty} c_n f_n(x) e^{-\lambda_n t},$$

where the c_n's are arbitrary constants.

To determine the c_n's in terms of the initial temperature $h(x)$, we need a generalization of the theory of Fourier series. The key idea here is that the eigenspaces should be orthogonal with respect to an appropriate inner product. The inner product should be one which makes $\mathbf{L}$ like a symmetric matrix. To arrange this, the inner product that we need to use on V_0 is the one defined by the formula

$$\langle f, g \rangle = \int_a^b \rho(x)\sigma(x) f(x) g(x) dx.$$

Lemma. *With respect to this inner product, eigenfunctions corresponding to distinct eigenvalues are perpendicular.*

The proof hinges on the fact that

$$\langle \mathbf{L}(f), g \rangle = \langle f, \mathbf{L}(g) \rangle, \qquad \text{for } f, g \in V_0,$$

[4] For further discussion, see Boyce and DiPrima, *Elementary differential equations and boundary value problems*, Seventh edition, Wiley, New York, 2001.

so that if we thought of L as represented by a matrix, the matrix would be symmetric. This identity can be verified by integration by parts; indeed,

$$\langle \mathbf{L}(f), g\rangle = \int_a^b \rho(x)\sigma(x)\mathbf{L}(f)(x)g(x)dx = \int_a^b \frac{d}{dx}\left(K(x)\frac{df}{dx}(x)\right)g(x)$$
$$= -\int_a^b K(x)\frac{df}{dx}(x)\frac{dg}{dx}(x)dx = \cdots = \langle f, \mathbf{L}(g)\rangle,$$

where the steps represented by dots are just like the first steps, but in reverse order.

It follows that if $f_i(x)$ and $f_j(x)$ are eigenfunctions corresponding to distinct eigenvalues λ_i and λ_j, then

$$\lambda_i\langle f_i, f_j\rangle = \langle \mathbf{L}(f), g\rangle = \langle f, \mathbf{L}(g)\rangle = \lambda_j\langle f_i, f_j\rangle,$$

and hence

$$(\lambda_i - \lambda_j)\langle f_i, f_j\rangle = 0.$$

Since $\lambda_i - \lambda_j \neq 0$, we conclude that f_i and f_j are perpendicular with respect to the inner product $\langle\cdot,\cdot\rangle$, as claimed.

Thus to determine the c_n's, we can use exactly the same orthogonality techniques that we have used before. Namely, if we normalize the eigenfunctions so that they have unit length and are orthogonal to each other with respect to $\langle,\cdot,\cdot\rangle$, then

$$c_n = \langle h, f_n\rangle,$$

or equivalently, c_n is just the projection of h in the f_n direction.

Example. We consider the operator

$$\mathbf{L} = x\frac{d}{dx}\left(x\frac{d}{dx}\right),$$

which acts on the space V_0 of functions $f : [1, e^\pi] \to \mathbb{R}$ which vanish at the endpoints of the interval $[1, e^\pi]$. To solve the eigenvalue problem, we need to find the nontrivial solutions to

$$x\frac{d}{dx}\left(x\frac{df}{dx}\right)(x) = \lambda f(x), \qquad f(1) = 0 = f(e^\pi). \tag{4.33}$$

We could find these nontrivial solutions by using the techniques we have learned for treating Cauchy-Euler equations.

However, there is a simpler approach, based upon the technique of substitution. Namely, we make a change of variables $x = e^z$ and note that since

$$dx = e^z dz, \qquad \frac{d}{dx} = \frac{1}{e^z}\frac{d}{dz} \qquad \text{and hence} \qquad x\frac{d}{dx} = e^z\frac{1}{e^z}\frac{d}{dz} = \frac{d}{dz}.$$

Thus if we set

$$\tilde{f}(z) = f(x) = f(e^z),$$

the eigenvalue problem (4.33) becomes

$$\frac{d^2\tilde{f}}{dz^2}(z) = \lambda \tilde{f}(z), \qquad \tilde{f}(0) = 0 = \tilde{f}(\pi),$$

a problem which we have already solved. The nontrivial solutions are

$$\lambda_n = -n^2, \qquad \tilde{f}_n(z) = \sin nz, \qquad \text{where } n = 1, 2, \ldots.$$

Thus the eigenvalues for our original problem are

$$\lambda_n = -n^2, \qquad \text{for } n = 1, 2, \ldots,$$

and as corresponding eigenfunctions we can take

$$f_n(x) = \sin(n \log x),$$

where log denotes the natural or base e logarithm. The lemma implies that these eigenfunctions will be perpendicular with respect to the inner product $\langle \cdot, \cdot \rangle$, defined by

$$\langle f, g \rangle = \int_1^{e^\pi} \frac{1}{x} f(x) g(x) dx. \tag{4.34}$$

Exercises:

4.7.1. Show by direct integration that if $m \neq n$, the functions

$$f_m(x) = \sin(m \log x) \qquad \text{and} \qquad f_n(x) = \sin(n \log x)$$

are perpendicular with respect to the inner product defined by (4.34).

4.7.2. Find the function $u(x, t)$, defined for $1 \leq x \leq e^\pi$ and $t \geq 0$, which satisfies the following initial-value problem for a heat equation with variable coefficients:

$$\frac{\partial u}{\partial t} = x \frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x} \right), \quad u(1, t) = u(e^\pi, t) = 0,$$

$$u(x, 0) = 3 \sin(\log x) + 7 \sin(2 \log x) - 2 \sin(3 \log x).$$

4.7.3.a. Find the solution to the eigenvalue problem for the operator

$$\mathbf{L} = x \frac{d}{dx}\left(x \frac{d}{dx} \right) - 3,$$

which acts on the space V_0 of functions $f : [1, e^\pi] \to \mathbb{R}$ which vanish at the endpoints of the interval $[1, e^\pi]$.

b. Find the function $u(x, t)$, defined for $1 \leq x \leq e^\pi$ and $t \geq 0$, which satisfies the following initial-value problem for a heat equation with variable coefficients:

$$\frac{\partial u}{\partial t} = x \frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x} \right) - 3u, \quad u(1, t) = u(e^\pi, t) = 0,$$

$$u(x, 0) = 3 \sin(\log x) + 7 \sin(2 \log x) - 2 \sin(3 \log x).$$

4.8 Numerical solutions to the eigenvalue problem*

We can also apply Sturm-Liouville theory to study the motion of a string of variable mass density. We can imagine a violin string stretched out along the x-axis with endpoints at $x = 0$ and $x = 1$ covered with a layer of varnish which causes its mass density to vary from point to point. We could let

$$\rho(x) = \text{the mass density of the string at } x \text{ for } 0 \leq x \leq 1.$$

If the string is under constant tension T, its motion might be governed by the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho(x)} \frac{\partial^2 u}{\partial x^2}, \tag{4.35}$$

which would be subject to the Dirichlet boundary conditions

$$u(0,t) = 0 = u(1,t), \qquad \text{for all } t \geq 0. \tag{4.36}$$

It is natural to try to find the general solution to (4.35) and (4.36) by separation of variables, letting $u(x,t) = f(x)g(t)$ as usual. Substituting into (4.35) yields

$$f(x)g''(t) = \frac{T}{\rho(x)} f''(x)g(t), \quad \text{or} \quad \frac{g''(t)}{g(t)} = \frac{T}{\rho(x)} \frac{f''(x)}{f(x)}.$$

The two sides must equal a constant, denoted by λ, and the partial differential equation separates into two ordinary differential equations,

$$\frac{T}{\rho(x)} f''(x) = \lambda f(x), \qquad g''(t) = \lambda g(t).$$

The Dirichlet boundary conditions (4.36) yield $f(0) = 0 = f(1)$. Thus f must satisfy the eigenvalue problem

$$\mathbf{L}(f) = \lambda f, \quad f(0) = 0 = f(1), \qquad \text{where} \qquad \mathbf{L} = \frac{T}{\rho(x)} \frac{d^2}{dx^2}.$$

Although the names of the functions appearing in $\mathbf{L}$ are a little different than those used in the previous section, the same Theorem applies. Thus the eigenvalues of $\mathbf{L}$ are negative real numbers and each eigenspace is one-dimensional. Moreover, the eigenvalues can be arranged in a sequence

$$0 > \lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots,$$

with $\lambda_n \to -\infty$. Finally, every well-behaved function can be represented on $[a,b]$ as a convergent sum of eigenfunctions. If $f_1(x)$, $f_2(x)$, $\ldots$, $f_n(x)$, $\ldots$ are eigenfunctions corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$. Then the general solution to (4.35) and (4.36) is

$$u(x,t) = \sum_{n=0}^{\infty} f_n(x) \left[a_n \cos(\sqrt{-\lambda_n} t) + b_n \sin(\sqrt{-\lambda_n} t) \right],$$

where the a_n's and b_n's are arbitrary constants. Each term in this sum represents one of the modes of oscillation of the vibrating string.

In constrast to the case of constant density, it is usually not possible to find simple explicit eigenfunctions when the density varies. It is therefore usually necessary to use numerical methods.

The simplest numerical method is the one outlined in §4.3. For $0 \leq i \leq n$, we let $x_i = i/n$ and

$$u_i(t) = u(x_i, t) = \text{the displacement at } x_i \text{ at time } t.$$

Since $u_0(t) = 0 = u_n(t)$ by the boundary conditions, the displacement at time t is approximated by

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ u_{n-1}(t) \end{pmatrix},$$

a vector-valued function of one variable. The partial derivative

$$\frac{\partial^2 u}{\partial t^2} \quad \text{is approximated by} \quad \frac{d^2 \mathbf{u}}{dt^2},$$

and as we saw in §4.3, the partial derivative

$$\frac{\partial^2 u}{\partial x^2} \quad \text{is approximated by} \quad n^2 P\mathbf{u},$$

where

$$P = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & \cdots & -2 \end{pmatrix}.$$

Finally, the coefficient $(T/\rho(x))$ can be represented by the diagonal matrix

$$Q = \begin{pmatrix} T/\rho(x_1) & 0 & 0 & \cdots & 0 \\ 0 & T/\rho(x_2) & 0 & \cdots & 0 \\ 0 & 0 & T/\rho(x_3) & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & \cdots & T/\rho(x_{n-1}) \end{pmatrix}.$$

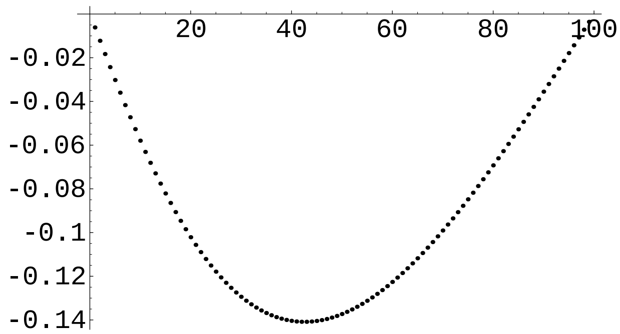

Figure 4.1: Shape of the lowest mode when $\rho = 1/(x + .1)$.

Putting all this together, we find that our wave equation with variable mass density is approximated by a second order homogeneous linear system of ordinary differential equations

$$\frac{d^2\mathbf{u}}{dt^2} = A\mathbf{u}, \qquad \text{where} \qquad A = n^2 QP.$$

The eigenvalues of low absolute value are approximated by the eigenvalues of A, while the eigenfunctions representing the lowest frequency modes of oscillation are approximated eigenvectors corresponding to the lowest eigenvalues of A.

For example, we could ask the question: What is the shape of the lowest mode of oscillation in the case where $\rho(x) = 1/(x+.1)$? To answer this question, we could utilize the following Mathematica program:

```
n := 100; rho[x_] := 1/(x + .1);
m := Table[Max[2-Abs[i-j],0], { i,n-1 } ,{ j,n-1 } ];
p := m - 4 IdentityMatrix[n-1];
q := DiagonalMatrix[Table[(1/rho[i/n]), { i,1,n-1 } ]];
a := n∧2 q.p; eigenvec = Eigenvectors[N[a]];
ListPlot[eigenvec[[n-1]]]
```

If we run this program we obtain a graph of the shape of the lowest mode, as shown in Figure 4.1. Note that instead of approximating a sine curve, our numerical approximation to the lowest mode tilts somewhat to the left.

Chapter 5

PDE's in Higher Dimensions

5.1 The three most important linear partial differential equations

In higher dimensions, the three most important linear partial differential equations are Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

the heat equation

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right),$$

and the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right),$$

where c is a nonzero constant, and u is a function of the space variables x, y and z, as well as of time t, in the second two cases. Note that all three equations can be written in terms of the *Laplace operator*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

as

$$\Delta u = 0, \quad \frac{\partial u}{\partial t} = c^2 \Delta u \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = c^2 \Delta u.$$

These three basic partial differential equations and their close relatives arise in many applications. One first develops techniques for studying them, and then extends those techniques to closely related equations.

Each of these three basic partial differential equations is *homogeneous linear*, that is each term contains u or one of its derivatives to the first power. This ensures that the principle of superposition will hold,

$$u_1 \text{ and } u_2 \text{ solutions} \quad \Rightarrow \quad c_1u_1 + c_2u_2 \text{ is a solution},$$

for any choice of constants c_1 and c_2. The principle of superposition is essential if we want to apply separation of variables and Fourier analysis techniques.

In the first few sections of this chapter, we will derive these partial differential equations in several contexts from physics. We will begin by using the divergence theorem to derive the heat equation, which in turn reduces in the steady-state case to Laplace's equation. We then present two derivations of the wave equation, one for vibrating membranes and one for sound waves. Exactly the same wave equation also describes electromagnetic waves, gravitational waves, or water waves in a linear approximation. It is remarkable that the principles developed to solve the three basic linear partial differential equations can be applied in many contexts.

In a few cases, it is possible to find explicit solutions to these partial differential equations under the simplest boundary conditions. For example, the general solution to the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

for the vibrations of an infinitely long string, was determined by d'Alembert to be

$$u(x,t) = f(x+ct) + g(x-ct),$$

where f and g are arbitrary well-behaved functions of a single variable.

Slightly more complicated cases can be treated by the technique of "separation of variables," together with Fourier analysis, as we saw in the previous chapter for partial differential equations with one space dimension. Separation of variables reduces these partial differential equations to linear ordinary differential equations, sometimes with variable coefficients. For example, to find the explicit solution to the heat equation in a circular room, we will see that it is necessary to solve Bessel's equation, while to solve Laplace's equation in the spherically symmetric case it is necessary to solve Legendre's equation.

The simplest nonlinear partial differential equations can sometimes be treated by the process of linearization. In the most complicated cases, one must resort to numerical methods, together with sufficient theory to understand the qualitative behaviour of solutions.

In the rest of this section, we consider a simple example, the equation governing heat flow through a region of (x,y,z)-space, under the assumption that no heat is being created or destroyed within the region. Let

$$u(x,y,z,t) = \text{temperature at } (x,y,z) \text{ at time } t.$$

If $\sigma(x,y,z)$ is the specific heat at the point (x,y,z) and $\rho(x,y,z)$ is the density of the medium at (x,y,z), then the heat within a given region D in

(x, y, z)-space is given by the formula

$$\text{Heat within } D = \iiint_D \rho(x, y, z)\sigma(x, y, z)u(x, y, z, t)dxdydz.$$

If no heat is being created or destroyed within D, then by conservation of energy, the rate at which heat leaves D equals minus the rate of change of heat within D, which is

$$-\frac{d}{dt}\left[\iiint_D \rho\sigma u dxdydz\right] = -\iiint_D \rho\sigma\frac{\partial u}{\partial t}dxdydz, \tag{5.1}$$

by differentiating under the integral sign.

On the other hand, heat flow can be represented by a vector field $\mathbf{F}(x, y, z, t)$ which points in the direction of greatest decrease of temperature,

$$\mathbf{F}(x, y, z, t) = -\kappa(x, y, z)(\nabla u)(x, y, z, t),$$

where $\kappa(x, y, z)$ is the so-called *thermal conductivity* of the medium at (x, y, z). Thus the rate at which heat leaves the region D through a small region in its boundary of area dA is

$$-(\kappa\nabla u) \cdot \mathbf{N}dA,$$

where $\mathbf{N}$ is the unit normal which points out of D. The total rate at which heat leaves D is given by the flux integral

$$-\iint_{\partial \mathbf{D}} (\kappa\nabla u) \cdot \mathbf{N}dA,$$

where ∂D is the surface bounding D. It follows from the divergence theorem that

$$\text{Rate at which heat leaves } D = -\iiint_D \nabla \cdot (\kappa\nabla u)dxdydz. \tag{5.2}$$

From formulae (5.1) and (5.2), we conclude that

$$\iiint_D \rho\sigma\frac{\partial u}{\partial t}dxdydz = \iiint_D \nabla \cdot (\kappa\nabla u)dxdydz.$$

This equation is true for *all* choices of the region D, so the integrands on the two sides must be equal:

$$\rho(x, y, z)\sigma(x, y, z)\frac{\partial u}{\partial t}(x, y, z, t) = \nabla \cdot (\kappa(x, y, z)\nabla u(x, y, z, t)).$$

Thus we finally obtain the *heat equation*

$$\frac{\partial u}{\partial t} = \frac{1}{\rho(x, y, z)\sigma(x, y, z)}\nabla \cdot (\kappa(x, y, z)(\nabla u)).$$

In the special case where the region D is *homogeneous*, i.e. its properties are the same at every point, $\rho(x,y,z)$, $\sigma(x,y,z)$ and $\kappa(x,y,z)$ are constants, and the heat equation becomes

$$\frac{\partial u}{\partial t} = \frac{\kappa}{\rho\sigma}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right].$$

If we wait long enough, so that the temperature is no longer changing, the "steady-state" temperature $u(x,y,z)$ must satisfy *Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

If the temperature is independent of z, the function $u(x,y) = u(x,y,z)$ must satisfy the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Exercises:

5.1.1. For which of the following differential equations is it true that the superposition principle holds?

$$\text{a.}\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

$$\text{b.}\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + u = 0.$$

$$\text{c.}\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + u^2 = 0.$$

$$\text{d.}\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = e^x.$$

$$\text{e.}\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = e^x u.$$

Explain your answers.

5.1.2. Suppose that a chemical reaction creates heat at the rate

$$\lambda(x,y,z)u(x,y,z,t) + \nu(x,y,z),$$

per unit volume. Show that in this case the equation governing heat flow is

$$\rho(x,y,z)\sigma(x,y,z)\frac{\partial u}{\partial t} = \lambda(x,y,z)u(x,y,z,t) + \nu(x,y,z) + \nabla \cdot (\kappa(x,y,z)(\nabla u)).$$

5.2 The Dirichlet problem

The reader will recall that the space of solutions to a homogeneous linear second order *ordinary* differential equation, such as

$$\frac{d^2u}{dt^2} + p\frac{du}{dt} + qu = 0$$

is two-dimensional, a particular solution being determined by two constants. By contrast, the space of solutions to Laplace's partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{5.3}$$

is *infinite*-dimensional. For example, the function $u(x,y) = x^3 - 3xy^2$ is a solution to Laplace's equation, because

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial^2 u}{\partial x^2} = 6x,$$

$$\frac{\partial u}{\partial y} = -6xy, \quad \frac{\partial^2 u}{\partial y^2} = -6x,$$

and hence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0.$$

Similarly,

$$u(x,y) = 7, \quad u(x,y) = x^4 - 6x^2y^2 + y^4, \quad \text{and} \quad u(x,y) = e^x \sin y$$

are solutions to Laplace's equation. A solution to Laplace's equation is called a *harmonic* function. It is not difficult to construct infinitely many linearly independent harmonic functions of two variables.

To pick out a particular solution to Laplace's equation, we need boundary conditions which will impose infinitely many constraints. To see what boundary conditions are natural to impose, we need to think of a physical problem which leads to Laplace's equation. Suppose that $u(x,y)$ represents the steady-state distribution of temperature throughout a uniform slab in the shape of a bounded region D in the (x,y)-plane. If we specify the temperature on the boundary ∂D of the region, say by setting up heaters and refrigerators controlled by thermostats along the boundary, we might expect that the temperature inside the room would be uniquely determined. We need infinitely many heaters and refrigerators because there are infinitely many points on the boundary. Specifying the temperature at each point of the boundary imposes infinitely many constraints on the harmonic function $u : D \to \mathbb{R}$ which realizes the steady-state temperature within D.

The Dirichlet Problem for Laplace's Equation. *Let D be a bounded region in the (x,y)-plane which is bounded by a continuous curve ∂D, and let*

$\phi : \partial D \to \mathbb{R}$ be a continuous function. Find a harmonic function $u : D \to \mathbb{R}$ such that

$$u(x, y) = \phi(x, y), \qquad \text{for } (x, y) \in \partial D.$$

Our physical intuition suggests that the Dirichlet problem will always have a unique solution. This is proven mathematically for many choices of boundary in more advanced texts on complex variables and partial differential equations. The mathematical proof that a unique solution exists provides evidence that the mathematical model we have constructed for heat flow makes good mathematical sense, providing evidence that the model may be valid.

Our goal here is to find the explicit solutions in the case where the region D is sufficiently simple. Indeed, suppose that

$$D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq a, 0 \leq y \leq b\}.$$

Note that the boundary ∂D of D consists of four line segments. Suppose, moreover that the function $\phi : \partial D \to \mathbb{R}$ vanishes on three sides of ∂D, so that

$$\phi(0, y) = \phi(a, y) = \phi(x, 0) = 0,$$

while

$$\phi(x, b) = f(x),$$

where $f(x)$ is a given continuous function which vanishes when $x = 0$ and $x = a$.

In this case, we seek a function $u(x, y)$, defined for $0 \leq x \leq a$ and $0 \leq y \leq b$, such that

1. $u(x, y)$ satisfies Laplace's equation
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{5.4}$$
2. $u(x, y)$ satisfies the homogeneous boundary conditions $u(0, y) = u(a, y) = u(x, 0) = 0$.
3. $u(x, y)$ satisfies the nonhomogeneous boundary condition $u(x, b) = f(x)$, where $f(x)$ is a given function.

Note that the Laplace equation itself and the homogeneous boundary conditions satisfy the superposition principle—this means that if u_1 and u_2 satisfy these conditions, so does $c_1 u_1 + c_2 u_2$, for any choice of constants c_1 and c_2.

Our method for solving the Dirichlet problem consists of two steps:

Step I. We find all of the solutions to Laplace's equation together with the homogeneous boundary conditions which are of the special form

$$u(x, y) = X(x)Y(y).$$

By the superposition principle, an arbitrary linear superposition of these solutions will still be a solution.

Step II. We find the particular solution which satisfies the nonhomogeneous boundary condition by Fourier analysis.

To carry out Step I, we substitute $u(x,y) = X(x)Y(y)$ into Laplace's equation (5.3) and obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Next we separate variables, putting all the functions involving x on the left, all the functions involving y on the right:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The left-hand side of this equation does not depend on y, while the right-hand side does not depend on x. Hence neither side can depend upon either x or y. In other words, the two sides must equal a constant, which we denote by λ, and call the separating constant, as before. Our equation now becomes

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda,$$

which separates into two ordinary differential equations,

$$X''(x) = \lambda X(x), \tag{5.5}$$

and

$$Y''(y) = -\lambda Y(y). \tag{5.6}$$

The homogeneous boundary condition $u(0,y) = u(a,y) = 0$ imply that

$$X(0)Y(y) = X(a)Y(y) = 0.$$

If $Y(y)$ is not identically zero,

$$X(0) = X(a) = 0.$$

Thus we need to find the nontrivial solutions to a boundary value problem for an ordinary differential equation:

$$X''(x) = \frac{d^2}{dx^2}(X(x)) = \lambda X(x), \quad X(0) = 0 = X(a),$$

which we recognize as the eigenvalue problem for the differential operator

$$\mathbf{L} = \frac{d^2}{dx^2}$$

acting on the space V_0 of functions which vanish at 0 and a. We have seen before that the only nontrivial solutions to equation (5.5) are constant multiples of

$$X(x) = \sin(n\pi x/a), \quad \text{with} \quad \lambda = -(n\pi/a)^2, \quad n = 1, 2, 3, \ldots.$$

For each of these solutions, we need to find a corresponding $Y(y)$ solving equation (5.6),

$$Y''(y) = -\lambda Y(y),$$

where $\lambda = -(n\pi/a)^2$, together with the boundary condition $Y(0) = 0$. The differential equation has the general solution

$$Y(y) = A\cosh(n\pi y/a) + B\sinh(n\pi y/a),$$

where A and B are constants of integration, and the boundary condition $Y(0) = 0$ implies that $A = 0$. Thus we find that the nontrivial product solutions to Laplace's equation together with the homogeneous boundary conditions are constant multiples of

$$u_n(x, y) = \sin(n\pi x/a)\sinh(n\pi y/a).$$

The *general solution* to Laplace's equation with these boundary conditions is a general superposition of these product solutions:

$$u(x, y) = B_1\sin(\pi x/a)\sinh(\pi y/a) + B_2\sin(2\pi x/a)\sinh(2\pi y/a) + \ldots. \tag{5.7}$$

To carry out Step II, we need to determine the constants $B_1, B_2, \ldots$ which occur in (5.7) so that

$$u(x, b) = f(x).$$

Substitution of $y = b$ into (5.7) yields

$$f(x) = B_1\sin(\pi x/a)\sinh(\pi b/a) + B_2\sin(2\pi x/a)\sinh(2\pi b/a) + \\ \ldots + B_k\sin(2\pi k/a)\sinh(k\pi b/a) + \ldots.$$

We see that $B_k\sinh(k\pi b/a)$ is the k-th coefficient in the Fourier sine series for $f(x)$.

For example, if $a = b = \pi$, and $f(x) = 3\sin x + 7\sin 2x$, then we must have

$$f(x) = B_1\sin(x)\sinh(\pi) + B_2\sin(2x)\sinh(2\pi),$$

and hence

$$B_1 = \frac{3}{\sinh(\pi)}, \quad B_2 = \frac{7}{\sinh(2\pi)}, \quad B_k = 0 \quad \text{for} \quad k = 3, \ldots.$$

Thus the solution to Dirichlet's problem in this case is

$$u(x, y) = \frac{3}{\sinh(\pi)}\sin x\sinh y + \frac{7}{\sinh(2\pi)}\sin 2x\sinh 2y.$$

Exercises:

5.2.1. Which of the following functions are harmonic?

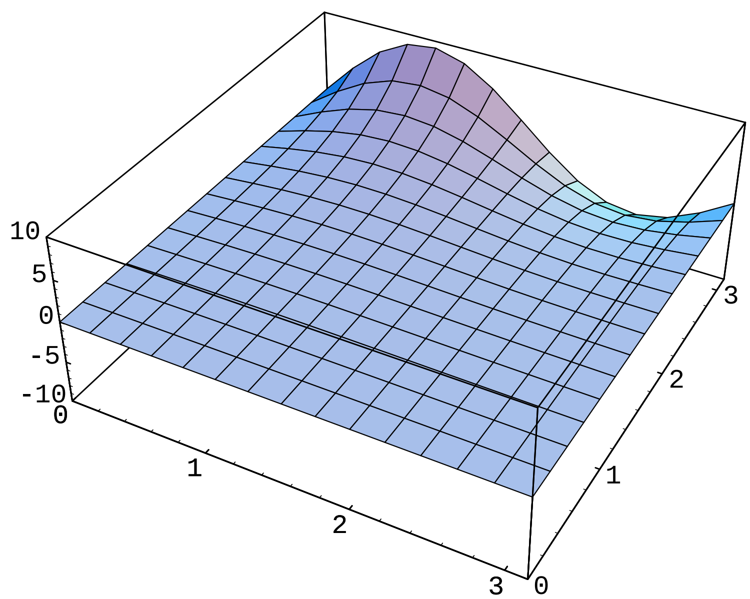

Figure 5.1: Graph of $u(x,y) = \frac{3}{\sinh(\pi)} \sin x \sinh y + \frac{7}{\sinh(2\pi)} \sin 2x \sinh 2y$.

a. $f(x,y) = x^2 + y^2$.

b. $f(x,y) = x^2 - y^2$.

c. $f(x,y) = e^x \cos y$.

d. $f(x,y) = x^4 - 6x^2y^2$.

5.2.2. a. Solve the following Dirichlet problem for Laplace's equation in a square region: Find $u(x,y), 0 \leq x \leq \pi, 0 \leq y \leq \pi$, such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(0,y) = u(\pi,y) = 0,$$
$$u(x,0) = 0, \qquad u(x,\pi) = \sin x - 2\sin 2x + 3\sin 3x.$$

b. Solve the following Dirichlet problem for Laplace's equation in the same square region: Find $u(x,y), 0 \leq x \leq \pi, 0 \leq y \leq \pi$, such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(0,y) = 0$$
$$u(\pi,y) = \sin 2y + 3\sin 4y, \qquad u(x,0) = 0 = u(x,\pi).$$

c. By adding the solutions to parts a and b together, find the solution to the

Dirichlet problem: Find $u(x, y), 0 \leq x \leq \pi, 0 \leq y \leq \pi$, such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(0, y) = 0, \qquad u(\pi, y) = \sin 2y + 3 \sin 4y,$$
$$u(x, 0) = 0, \qquad u(x, \pi) = \sin x - 2 \sin 2x + 3 \sin 3x.$$

5.2.3. Solve the following Dirichlet problem for Laplace's equation in a square region: Find $u(x, y), 0 \leq x \leq \pi, 0 \leq y \leq \pi$, such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(0, y) = 0,$$

$$u(\pi, y) = \begin{cases} y, & \text{for } 0 \leq y \leq \pi/2, \\ \pi - y, & \text{for } \pi/2 \leq y \leq \pi, \end{cases} \qquad u(x, 0) = 0 = u(x, \pi).$$

5.3 Initial value problems for heat equations

The physical interpretation behind the heat equation suggests that the following initial value problem should have a unique solution.

Let D be a bounded region in the (x, y)-plane which is bounded by a piecewise smooth curve ∂D, and let $h : D \to \mathbb{R}$ be a continuous function which vanishes on ∂D. Find a function $u(x, y, t)$ such that

1. u satisfies the heat equation
$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right). \tag{5.8}$$

2. u satisfies the "Dirichlet boundary condition" $u(x, y, t) = 0$, for $(x, y) \in \partial D$.

3. u satisfies the initial condition $u(x, y, 0) = h(x, y)$.

The first two of these conditions are homogeneous and linear, so our strategy is to treat them first by separation of variables, and then use Fourier analysis to satisfy the last condition.

In this section, we consider the special case where

$$D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq a, 0 \leq y \leq b\},$$

so that the Dirichlet boundary condition becomes

$$u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0.$$

In this case, separation of variables is done in two stages. First, we write

$$u(x, y, t) = f(x, y)g(t),$$

and substitute into (5.8) to obtain

$$f(x,y)g'(t) = c^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) g(t).$$

Then we divide by $c^2 f(x,y)g(t)$,

$$\frac{1}{c^2 g(t)} g'(t) = \frac{1}{f(x,y)} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right).$$

The left-hand side of this equation does not depend on x or y while the right hand side does not depend on t. Hence neither side can depend on x, y, or t, so both sides must be constant. If we let λ denote the constant, we obtain

$$\frac{1}{c^2 g(t)} g'(t) = \lambda = \frac{1}{f(x,y)} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right).$$

This separates into an ordinary differential equation

$$g'(t) = c^2 \lambda g(t), \tag{5.9}$$

and a partial differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \lambda f, \tag{5.10}$$

called the *Helmholtz equation*. The Dirichlet boundary condition yields

$$f(0,y) = f(a,y) = f(x,0) = f(x,b) = 0.$$

In the second stage of separation of variables, we set $f(x,y) = X(x)Y(y)$ and substitute into (5.10) to obtain

$$X''(x)Y(y) + X(x)Y''(y) = \lambda X(x)Y(y),$$

which yields

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \lambda,$$

or

$$\frac{X''(x)}{X(x)} = \lambda - \frac{Y''(y)}{Y(y)}.$$

The left-hand side depends only on x, while the right-hand side depends only on y, so both sides must be constant,

$$\frac{X''(x)}{X(x)} = \mu = \lambda - \frac{Y''(y)}{Y(y)}.$$

Hence the Helmholtz equation divides into two ordinary differential equations

$$X''(x) = \mu X(x), \quad Y''(y) = \nu Y(y), \quad \text{where} \quad \mu + \nu = \lambda.$$

The "Dirichlet boundary conditions" now become conditions on $X(x)$ and $Y(y)$:

$$X(0) = X(a) = 0, \quad Y(0) = Y(b) = 0.$$

The only nontrivial solutions are

$$X(x) = \sin(m\pi x/a), \quad \text{with} \quad \mu = -(m\pi/a)^2, \quad m = 1, 2, 3, \ldots,$$

and

$$Y(y) = \sin(n\pi y/b), \quad \text{with} \quad \nu = -(n\pi/b)^2, \quad n = 1, 2, 3, \ldots.$$

The corresponding solutions of the Helmholtz equation are

$$f_{mn}(x, y) = \sin(m\pi x/a)\sin(n\pi y/b), \quad \text{with} \quad \lambda_{mn} = -(m\pi/a)^2 - (n\pi/b)^2,$$

where m and n range over the natural numbers

$$N = \{1, 2, 3, \cdots\}.$$

We can rephrase in terms of linear algebra as follows: Let V_0 be the space of smooth functions $f : [0, a] \times [0, b] \to R$ such that

$$f(0, y) = f(a, y) = f(x, 0) = f(x, b) = 0.$$

We wish to find the values of $\lambda \in R$ for which the "eigenspace"

$$W_\lambda = \left\{ f \in V_0 : \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \lambda f \right\}$$

is not the zero subspace. The λ's for which W_λ is nonzero are called the *eigenvalues* of Δ on V_0, and the solutions to $\Delta f = \lambda f$ are called *eigenfunctions*. We have seen that these eigenvalues are

$$\lambda_{m,n} = -\left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2, \quad \text{with corresponding eigenfunctions}$$

$$f_{mn}(x, y) = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right).$$

For any given choices of $m, n \in N$, the corresponding solution to (5.9) is

$$g(t) = e^{-c^2\lambda_{mn}t} = e^{-c^2((m\pi/a)^2+(n\pi/b)^2)t}.$$

Hence for each choice of $m, n \in N$, we obtain a product solution to the heat equation with Dirichlet boundary condition:

$$u_{m,n}(x, y, t) = \sin(m\pi x/a)\sin(n\pi y/b)e^{-c^2((m\pi/a)^2+(n\pi/b)^2)t}.$$

It is natural to suspect that he general solution to the heat equation with Dirichlet boundary conditions is an arbitrary superposition of these product solutions,

$$u(x, y, t) = \sum_{m,n=1}^{\infty} b_{mn}\sin(m\pi x/a)\sin(n\pi y/b)e^{-c^2((m\pi/a)^2+(n\pi/b)^2)t}. \tag{5.11}$$

To see that this is the case, we need to be able to satisfy an arbitrary initial condition $u(x, y, 0) = h(x, y)$. It follows from (5.11) that $h(x, y)$ must satisfy

$$h(x,y) = \sum_{m,n=1}^{\infty} b_{mn} \sin(m\pi x/a) \sin(n\pi y/b). \tag{5.12}$$

For fixed choice of y, we can write

$$h(x,y) = \sum_{m=1}^{\infty} b_m(y) \sin(m\pi x/a),$$

$$\text{where} \qquad b_m(y) = \frac{2}{a} \int_0^a h(x,y) \sin(m\pi x/a) dx. \tag{5.13}$$

We can then write

$$b_m(y) = \sum_{n=1}^{\infty} b_{mn} \sin(n\pi y/b), \qquad \text{where} \qquad b_{mn} = \frac{2}{b} \int_0^b b_m(y) \sin(n\pi y/b) dy.$$

Substituting this into (5.13), we find that

$$h(x,y) = \sum_{m,n=1}^{\infty} b_{mn} \sin(m\pi x/a) \sin(n\pi y/b),$$

where

$$b_{mn} = \frac{2}{a}\frac{2}{b} \int_0^a \int_0^b h(x,y) \sin(m\pi x/a) \sin(m\pi y/b) dx dy. \tag{5.14}$$

We say that the b_{mn}'s are the coefficients of the *double Fourier sine series* of h.

Suppose, for example, that $c = 1$, $a = b = \pi$ and

$$h(x,y) = \sin x \sin y + 3 \sin 2x \sin y + 7 \sin 3x \sin 2y.$$

In this case, we do not need to carry out the integration indicated in (5.14) because comparison with (5.12) shows immediately that

$$b_{11} = 1, \quad b_{21} = 3, \quad b_{32} = 7,$$

and all the other b_{mn}'s must be zero. Thus the solution to the initial value problem in this case is

$$u(x,y,t) = \sin x \sin y e^{-2t} + 3 \sin 2x \sin y e^{-5t} + 7 \sin 3x \sin 2y e^{-13t}.$$

Here is another example. Suppose that $a = b = \pi$ and

$$h(x,y) = p(x)q(y),$$

where

$$p(x) = \begin{cases} x, & \text{for } 0 \le x \le \pi/2, \\ \pi - x, & \text{for } \pi/2 \le x \le \pi, \end{cases} \qquad q(y) = \begin{cases} y, & \text{for } 0 \le y \le \pi/2, \\ \pi - y, & \text{for } \pi/2 \le y \le \pi. \end{cases}$$

In this case,

$$\begin{aligned}
b_{mn} &= \left(\frac{2}{\pi}\right)^2 \int_0^\pi \left[\int_0^\pi p(x)q(y)\sin mx dx\right] \sin ny dy \\
&= \left(\frac{2}{\pi}\right)^2 \left[\int_0^\pi p(x)\sin mx dx\right]\left[\int_0^\pi q(y)\sin ny dy\right] \\
&= \left(\frac{2}{\pi}\right)^2 \left[\int_0^{\pi/2} x\sin mx dx + \int_{\pi/2}^\pi (\pi - x)\sin mx dx\right] \\
&\qquad \left[\int_0^{\pi/2} y\sin ny dy + \int_{\pi/2}^\pi (\pi - y)\sin ny dy\right].
\end{aligned}$$

The integration can be carried out just like we did in Section 3.3 to yield

$$\begin{aligned}
b_{mn} &= \left(\frac{2}{\pi}\right)^2 \left[\frac{2}{m^2}\sin(m\pi/2)\right]\left[\frac{2}{n^2}\sin(n\pi/2)\right] \\
&= \frac{16}{\pi^2}\left[\frac{1}{m^2}\frac{1}{n^2}\sin(m\pi/2)\sin(n\pi/2)\right].
\end{aligned}$$

Thus in this case, we see that

$$h(x,y) = \frac{16}{\pi^2}\sum_{m,n=1}^\infty \left[\frac{1}{m^2}\frac{1}{n^2}\sin(m\pi/2)\sin(n\pi/2)\right]\sin mx \sin my,$$

and hence

$$u(x,y,t) = \frac{16}{\pi^2}\sum_{m,n=1}^\infty \left[\frac{1}{m^2}\frac{1}{n^2}\sin(m\pi/2)\sin(n\pi/2)\right]\sin mx \sin my e^{-(m^2+n^2)t}.$$

Exercises:

5.3.1. Solve the following initial value problem for the heat equation in a square region: Find $u(x,y,t)$, where $0 \le x \le \pi, 0 \le y \le \pi$ and $t \ge 0$, such that

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \\
& u(x,0,t) = u(x,\pi,t) = u(0,y,t) = u(\pi,y,t) = 0, \\
& \qquad u(x,y,0) = 2\sin x \sin y + 5\sin 2x \sin y.
\end{aligned}$$

You may assume that the nontrivial solutions to the eigenvalue problem

$$\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = \lambda f(x,y), \quad f(x,0) = f(x,\pi) = f(0,y) = f(\pi,y) = 0,$$

are of the form

$$\lambda = -m^2 - n^2, \qquad f(x,y) = b_{mn}\sin mx \sin ny,$$

for $m = 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$, where b_{mn} is a constant.

5.3.2. Solve the following initial value problem for the heat equation in a square region: Find $u(x, y, t)$, where $0 \leq x \leq \pi, 0 \leq y \leq \pi$ and $t \geq 0$, such that

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$
$$u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0,$$
$$u(x, y, 0) = 2(\sin x)y(\pi - y).$$

5.3.3. Solve the following initial value problem in a square region: Find $u(x, y, t)$, where $0 \leq x \leq \pi, 0 \leq y \leq \pi$ and $t \geq 0$, such that

$$\frac{1}{2t+1}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$
$$u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0,$$
$$u(x, y, 0) = 2 \sin x \sin y + 3 \sin 2x \sin y.$$

5.3.4. Solve the following initial value problem in a square region: Find $u(x, y, t)$, where $0 \leq x \leq \pi, 0 \leq y \leq \pi$ and $t \geq 0$, such that

$$(t+1)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$
$$u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0,$$
$$u(x, y, 0) = \sin x \sin 2y + 5 \sin 2x \sin 2y.$$

5.3.5. Solve the following initial value problem in a square region: Find $u(x, y, t)$, where $0 \leq x \leq \pi, 0 \leq y \leq \pi$ and $t \geq 0$, such that

$$\frac{\partial u}{\partial t} + 3u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$
$$u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0,$$
$$u(x, y, 0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m!n!} \sin mx \sin ny.$$

5.3.6. Find the general solution to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

subject to the boundary conditions

$$u(x, 0, t) = u(0, y, t) = u(\pi, y, t) = 0,$$
$$u(x, \pi, t) = \sin x - 2 \sin 2x + 3 \sin 3x.$$

5.3.7. Find the solutions to the eigenvalue problem with Neumann boundary conditions,

$$\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = \lambda f(x,y),$$
$$\frac{\partial f}{\partial y}(x,0) = \frac{\partial f}{\partial y}(x,\pi) = \frac{\partial f}{\partial x}(0,y) = \frac{\partial f}{\partial x}(\pi,y) = 0,$$

which corresponds to insulated boundaries.

5.3.8. Solve the following initial value problem for the heat equation in a square region: Find $u(x,y,t)$, where $0 \leq x \leq \pi, 0 \leq y \leq \pi$ and $t \geq 0$, such that

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$
$$\frac{\partial u}{\partial y}(x,0,t) = \frac{\partial u}{\partial yx}(x,\pi,t) = \frac{\partial u}{\partial x}(0,y,t) = \frac{\partial u}{\partial x}(\pi,y,t) = 0,$$
$$u(x,y,0) = 2 + 7\cos x + 4\cos y + 12\cos x\cos y.$$

Hint: Use the result of Problem 5.3.7.

5.4 Two derivations of the wave equation

We have already seen how the one-dimensional wave equation describes the motion of a vibrating string. In this section we will show that the motion of a vibrating membrane is described by the two-dimensional wave equation, while sound waves are described by a three-dimensional wave equation. In fact, we will see that sound waves arise from "linearization" of the nonlinear equations of fluid mechanics.

The vibrating membrane. Suppose that a homogeneous membrane is fastened down along the boundary of a region D in the (x,y)-plane. Suppose, moreover, that a point on the membrane can move only in the vertical direction, and let $u(x,y,t)$ denote the height of the point with coordinates (x,y) at time t.

If ρ denotes the density of the membrane (assumed to be constant), then by Newton's second law of motion, the force $\mathbf{F}$ acting on a small rectangular piece of the membrane located at (x,y) with sides of length dx and dy is given by the expression

$$\mathbf{F} = \rho\frac{\partial^2 u}{\partial t^2}(x,y)dxdy\mathbf{k},$$

where $\mathbf{k}$ is the unit-length vector which points straight up. Suppose the force displaces the membrane from a given position $u(x,y)$ to a new position

$$u(x,y) + \eta(x,y),$$

where $\eta(x, y)$ and its derivatives are very small. Then the total work performed by the force $\mathbf{F}$ on the rectangular piece of membrane will be

$$\mathbf{F} \cdot \eta(x, y)\mathbf{k} = \eta(x, y)\rho \frac{\partial^2 u}{\partial t^2}(x, y) dx dy.$$

Integrating over the membrane yields an expression for the total work performed when the membrane moves through the displacement η:

$$\text{Work} = \int \int_D \eta(x, y)\rho \frac{\partial^2 u}{\partial t^2}(x, y) dx dy. \tag{5.15}$$

On the other hand, the potential energy stored in the membrane is proportional to the extent to which the membrane is stretched. Just as in the case of the vibrating string, this stretching is approximated by the integral

$$\text{Potential energy} = \frac{T}{2} \iint_D \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx dy,$$

where T is a constant, called the *tension* in the membrane. Replacing u by $u + \eta$ in this integral yields

$$\begin{aligned} \text{New potential energy} = \frac{T}{2} \iint_D \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx dy \\ + T \iint_D \left[\left(\frac{\partial u}{\partial x} \right) \left(\frac{\partial \eta}{\partial x} \right) + \left(\frac{\partial u}{\partial y} \right) \left(\frac{\partial \eta}{\partial y} \right) \right] dx dy \\ + \frac{T}{2} \iint_D \left[\left(\frac{\partial \eta}{\partial x} \right)^2 + \left(\frac{\partial \eta}{\partial y} \right)^2 \right] dx dy. \end{aligned}$$

If we neglect the last term in this expression (which is justified if η and its derivatives are assumed to be small), we find that

$$\text{New potential energy} - \text{Old potential energy} = \iint_D T \nabla u \cdot \nabla \eta dx dy.$$

It follows from the divergence theorem in the plane and the fact that η vanishes on the boundary ∂D that

$$\iint_D T \nabla u \cdot \nabla \eta dx dy + \iint_D T \eta \nabla \cdot \nabla u dx dy = \int_{\partial D} T \eta \nabla u \cdot \mathbf{N} ds = 0,$$

and hence

$$\text{Change in potential} = - \iint_D \eta(x, y) T (\nabla \cdot \nabla u)(x, y) dx dy. \tag{5.16}$$

The work performed must be minus the change in potential energy, so it follows from (5.15) and (5.16) that

$$\iint_D \eta(x,y)\rho\frac{\partial^2 u}{\partial t^2}(x,y)dxdy = \iint_D \eta(x,y)T(\nabla\cdot\nabla u)(x,y)dxdy.$$

Since this equation holds for all choices of η, it follows that

$$\rho\frac{\partial^2 u}{\partial t^2} = T\nabla\cdot\nabla u,$$

which simplifies to

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla\cdot\nabla u, \quad \text{where} \quad c^2 = \frac{T}{2},$$

or equivalently

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

This is just the wave equation.

The equations of a perfect gas and sound waves. Next, we will describe *Euler's equations* for a perfect gas in (x,y,z)-space.[1] Euler's equations are expressed in terms of the quantities,

$$\mathbf{v}(x,y,z,t) = (\text{velocity of the gas at } (x,y,z) \text{ at time } t),$$

$$\rho(x,y,z,t) = (\text{density at } (x,y,z) \text{ at time } t),$$

$$p(x,y,z,t) = (\text{pressure at } (x,y,z) \text{ at time } t).$$

The first of the Euler equations is the equation of continuity,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0. \tag{5.17}$$

We recall from Chapter 1 that we can derive this equation by representing the fluid flow by the vector field

$$\mathbf{F} = \rho\mathbf{v},$$

so that the surface integral

$$\iint_{\mathbf{S}} \mathbf{F}\cdot\mathbf{N}dA$$

represents the rate at which the fluid is flowing across $\mathbf{S}$ in the direction of $\mathbf{N}$. We assume that no fluid is being created or destroyed. Then the rate of change of the mass of fluid within D is given by two expressions,

$$\iiint_D \frac{\partial\rho}{\partial t}(x,y,z,t)dxdydz \qquad \text{and} \qquad -\iint_{\mathbf{S}} \mathbf{F}\cdot\mathbf{N}dA,$$

[1] For a complete derivation of these equations, see Chapter 9 of Alexander Fetter and John Walecka, *Theoretical mechanics of particles and continua*, McGraw-Hill, New York, 1980, or Chapter 1 of Alexandre Chorin and Jerrold Marsden, *A mathematical introduction to fluid mechanics*, third edition, Springer, 1993.

which must be equal. It follows from the divergence theorem that the second of these expressions is

$$-\iiint_D \nabla \cdot \mathbf{F}(x, y, z, t) dxdydz,$$

and hence

$$\iiint_D \frac{\partial \rho}{\partial t} dxdydz = -\iiint_D \nabla \cdot \mathbf{F} dxdydz.$$

Since this equation must hold for *every* region D in (x, y, z)-space, we conclude that the integrands must be equal,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{F} = -\nabla \cdot (\rho \mathbf{v}),$$

which is just the equation of continuity.

The second of the Euler equations is simply Newton's second law of motion,

$$\text{(mass density)(acceleration)} = \text{(force density)}.$$

We make an assumption that the only force acting on a fluid element is due to the pressure, an assumption which is not unreasonable in the case of a perfect gas. In this case, it turns out that the pressure is defined in such a way that the force acting on a fluid element is minus the gradient of pressure:

$$\text{Force} = -\nabla p(x(t), y(t), z(t), t). \tag{5.18}$$

The familiar formula Force = Mass × Acceleration then yields

$$\rho \frac{d}{dt} (\mathbf{v}(x(t), y(t), z(t), t)) = -\nabla p(x(t), y(t), z(t), t).$$

It follows from the chain rule,

$$\begin{aligned} \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x} \frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y} \frac{dy}{dt} + \frac{\partial \mathbf{v}}{\partial z} \frac{dz}{dt} + \frac{\partial \mathbf{v}}{\partial t} \frac{dt}{dt} \\ = \left(\frac{\partial \mathbf{v}}{\partial x}, \frac{\partial \mathbf{v}}{\partial y}, \frac{\partial \mathbf{v}}{\partial z} \right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) + \frac{\partial \mathbf{v}}{\partial t}, \end{aligned}$$

that we can rewrite this equation as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho} \nabla p. \tag{5.19}$$

Note that this equation is nonlinear because of the term $(\mathbf{v} \cdot \nabla)\mathbf{v}$, which is quadratic in $\mathbf{v}$.

To finish the Euler equations, we need an *equation of state*, which relates pressure and density. The equation of state could be determined by experiment, the simplest equation of state being

$$p = a^2 \rho^\gamma, \tag{5.20}$$

where a^2 and γ are constants. (An ideal monatomic gas has this equation of state with $\gamma = 5/3$.)

In summary the Euler equations (5.17), (5.19), and (5.20) for a perfect gas are

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}), \quad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p. \quad p = a^2 \rho^\gamma.$$

These equations are nonlinear, and hence quite difficult to solve. However, one explicit solution is the case where the fluid is motionless,

$$\rho = \rho_0, \quad p = p_0, \quad \mathbf{v} = 0,$$

where ρ_0 and p_0 satisfy

$$p_0 = a^2 \rho_0^\gamma.$$

Linearizing Euler's equations near this explicit solution gives rise to a linear partial differential equation which governs propagation of sound waves.

Let us write

$$\rho = \rho_0 + \rho', p = p_0 + p', \mathbf{v} = \mathbf{v}',$$

where ρ', p' and $\mathbf{v}'$ are so small that their squares can be ignored, and in particular, the term $(\mathbf{v} \cdot \nabla)\mathbf{v}$ in (5.19) can be dropped. Substitution into Euler's equations (5.17), (5.19) and (5.20) then yields

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot (\mathbf{v}') = 0,$$

$$\frac{\partial \mathbf{v}'}{\partial t} = -\frac{1}{\rho_0}\nabla p',$$

and

$$p' = [a^2 \gamma (\rho_0)^{(\gamma - 1)}]\rho' = c^2 \rho',$$

where c^2 is a new constant. It follows from these three equations that

$$\frac{\partial^2 \rho'}{\partial t^2} = -\rho_0 \left(\nabla \cdot \frac{\partial \mathbf{v}'}{\partial t} \right) = \nabla \cdot \nabla p' = c^2 \nabla \cdot \nabla \rho' = c^2 \Delta u.$$

Thus ρ' must satisfy the three-dimensional wave equation

$$\frac{\partial^2 \rho'}{\partial t^2} = c^2 \nabla \cdot \nabla \rho' = c^2 \left(\frac{\partial^2 \rho'}{\partial x^2} + \frac{\partial^2 \rho'}{\partial y^2} + \frac{\partial^2 \rho'}{\partial z^2} \right) = c^2 \Delta u. \tag{5.21}$$

If the sound wave ρ' is independent of z, (5.21) reduces to

$$\frac{\partial^2 \rho'}{\partial t^2} = c^2 \left(\frac{\partial^2 \rho'}{\partial x^2} + \frac{\partial^2 \rho'}{\partial y^2} \right),$$

exactly the same equation that we obtained for the vibrating membrane.

Remark: The notion of linearization is extremely powerful because it enables us to derive information on the behavior of solutions to the nonlinear Euler equations, which are extremely difficult to solve except under very special circumstances.

The Euler equations for a perfect gas and the closely related Navier-Stokes equations for an incompressible fluid such as water form basic models for fluid mechanics. In the case of incompressible fluids, the density is constant, so no equation of state is assumed. To allow for viscosity, one adds an additional term to the expression (5.18) for the force acting on a fluid element:

$$\text{Force} = \nu(\Delta \mathbf{v})(x, y, z, t) - \nabla p(x, y, z, t).$$

Here the Laplace operator Δ is applied componentwise and ν is a constant, called the viscosity of the fluid. The equations used by Navier and Stokes to model an incompressible viscous fluid (with ρ constant) are then

$$\nabla \cdot \mathbf{v} = 0, \qquad \rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = \nu \Delta \mathbf{v} - \nabla p.$$

It is remarkable that these equations, so easily expressed, are so difficult to solve. Indeed, the equations of fluid mechanics, the Euler and Navier-Stokes equations, form the basis for one of the seven Millenium Prize Problems, singled out by the Clay Mathematics Institute as central problems for mathematics at the turn of the century. If you can show that under reasonable initial conditions, the Navier-Stokes equations possess a unique well-behaved solution, you may be able to win one million dollars. To find more details on the prize offered for a solution, you can consult the web address: http://www.claymath.org/millennium/

Exercise:

5.4.1. Find the linearizations of the following partial differential equations at the solution $u \equiv 0$:

$$\text{a.} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = u^2.$$

$$\text{b.} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \sin u.$$

$$\text{c.} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + u \frac{\partial u}{\partial x} = 0.$$

5.5 Initial value problems for wave equations

The most natural initial value problem for the wave equation is the following: Let D be a bounded region in the (x, y)-plane which is bounded by a piecewise smooth curve ∂D, and let $h_1, h_2 : D \to \mathbb{R}$ be continuous functions. Find a function $u(x, y, t)$ such that

1. u satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right). \tag{5.22}$$

2. u satisfies the "Dirichlet boundary condition" $u(x,y,t) = 0$, for $(x,y) \in \partial D$.

3. u satisfies the initial condition

$$u(x,y,0) = h_1(x,y), \quad \frac{\partial u}{\partial t}(x,y,0) = h_2(x,y).$$

Solution of this initial value problem via separation of variables is very similar to the solution of the initial value problem for the heat equation which was presented in Section 5.3.

As before, let us suppose that $D = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le a, 0 \le y \le b\}$, so that the Dirichlet boundary condition becomes

$$u(0,y,t) = u(a,y,t) = u(x,0,t) = u(x,b,t) = 0.$$

We write

$$u(x,y,t) = f(x,y)g(t),$$

and substitute into (5.22) to obtain

$$f(x,y)g''(t) = c^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) g(t).$$

Then we divide by $c^2 f(x,y)g(t)$,

$$\frac{1}{c^2 g(t)} g''(t) = \frac{1}{f(x,y)} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right).$$

Once again, we conclude that both sides must be constant. If we let λ denote the constant, we obtain

$$\frac{1}{c^2 g(t)} g''(t) = \lambda = \frac{1}{f(x,y)} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right).$$

This separates into an ordinary differential equation

$$g''(t) = c^2 \lambda g(t), \tag{5.23}$$

and the Helmholtz equation

$$\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) = \lambda f. \tag{5.24}$$

The Dirichlet boundary condition becomes

$$f(0,y) = f(a,y) = f(x,0) = f(x,b) = 0.$$

The Helmholtz equation is solved in exactly the same way as before, the only nontrivial solutions being

$$f_{mn}(x,y) = \sin(m\pi x/a)\sin(n\pi y/b), \quad \text{with} \quad \lambda_{mn} = -(m\pi/a)^2 - (n\pi/b)^2.$$

The corresponding solution to (5.23) is

$$g(t) = A\cos(\omega_{mn}t) + B\sin(\omega_{mn}t), \qquad \text{where } \omega_{mn} = c\sqrt{-\lambda_{mn}}.$$

Thus we obtain a product solution to the wave equation with Dirichlet boundary conditions:

$$u(x,y,t) = \sin(m\pi x/a)\sin(n\pi y/b)[A\cos(\omega_{mn}t) + B\sin(\omega_{mn}t)].$$

The general solution to the wave equation with Dirichlet boundary conditions is a superposition of these product solutions,

$$u(x,y,t) = \sum_{m,n=1}^{\infty} \sin(m\pi x/a)\sin(n\pi y/b)[A_{mn}\cos(\omega_{mn}t) + B_{mn}\sin(\omega_{mn}t)].$$

The constants A_{mn} and B_{mn} are determined from the initial conditions.

The initial value problem considered in this section could represent the motion of a vibrating membrane. Just like in the case of the vibrating string, the motion of the membrane is a superposition of infinitely many modes, the mode corresponding to the pair (m,n) oscillating with frequency $\omega_{mn}/2\pi$. The lowest frequency of vibration or *fundamental frequency* is

$$\frac{\omega_{11}}{2\pi} = \frac{c}{2\pi}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2} = \frac{1}{2}\sqrt{\frac{T}{\rho}\left[\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2\right]}.$$

Exercises:

5.5.1 What happens to the frequency of the fundamental mode of oscillation of a vibrating rectangular membrane when the tension on the membrane is doubled? When the density of the membrane is doubled?

5.5.2. Solve the following initial value problem for a vibrating square membrane (where we have set $T = \rho = 1$): Find $u(x,y,t), 0 \le x \le \pi, 0 \le y \le \pi$, such that

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

$$u(x,0,t) = u(x,\pi,t) = u(0,y,t) = u(\pi,y,t) = 0,$$

$$u(x,y,0) = 3\sin x \sin y + 7\sin 2x \sin y, \qquad \frac{\partial u}{\partial t}(x,y,0) = 0.$$

5.5.3. Solve the following initial value problem for a vibrating square membrane: Find $u(x,y,t), 0 \le x \le \pi, 0 \le y \le \pi$, such that

$$\frac{\partial^2 u}{\partial t^2} = 4\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$

$$u(x,0,t) = u(x,\pi,t) = u(0,y,t) = u(\pi,y,t) = 0,$$

$$u(x,y,0) = 0, \quad \frac{\partial u}{\partial t}(x,y,0) = 2\sin x \sin y + 13 \sin 2x \sin y.$$

5.5.4. Solve the following initial value problem for a vibrating square membrane: Find $u(x,y,t), 0 \le x \le \pi, 0 \le y \le \pi$, such that

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

$$u(x,0,t) = u(x,\pi,t) = u(0,y,t) = u(\pi,y,t) = 0,$$

$$u(x,y,0) = p(x)q(y), \quad \frac{\partial u}{\partial t}(x,y,0) = 0,$$

where

$$p(x) = \begin{cases} x, & \text{for } 0 \le x \le \pi/2, \\ \pi - x, & \text{for } \pi/2 \le x \le \pi, \end{cases} \qquad q(y) = \begin{cases} y, & \text{for } 0 \le y \le \pi/2, \\ \pi - y, & \text{for } \pi/2 \le y \le \pi. \end{cases}$$

5.6 The Laplace operator in polar coordinates

In order to solve the heat equation over a circular plate, or to solve the wave equation for a vibrating circular drum, we need to express the Laplace operator in polar coordinates (r,θ). These coordinates are related to the standard Euclidean coordinates by the formulae

$$x = x(r,\theta) = r\cos\theta, \qquad y = y(r,\theta) = r\sin\theta.$$

The tool we need to express the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

in terms of polar coordinates is just the chain rule, which we studied earlier in the course. If $u(x,y)$ is a function of x and y, we can define a new function $\tilde{u}$ of r and θ by

$$\tilde{u}(r,\theta) = u(x(r,\theta), y(r,\theta)).$$

The chain rule then states that

$$\frac{\partial \tilde{u}}{\partial r} = \frac{\partial u}{\partial x}(x(r,\theta), y(r,\theta))\frac{\partial x}{\partial r}(r,\theta) + \frac{\partial u}{\partial y}(x(r,\theta), y(r,\theta))\frac{\partial y}{\partial r}(r,\theta).$$

The variables represented by the functions u and $\tilde{u}$ are the same (for example, both may represent temperature), so we usually drop the tilde and write u instead of $\tilde{u}$. This is commonly done, because otherwise we would run out of letters of the alphabet, and additional symbols such as tildes, in complicated problems.

Although straigtforward, the calculation of the Laplace operator in polar coordinates is somewhat lengthy. Since

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r\cos\theta) = \cos\theta, \quad \frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(r\sin\theta) = \sin\theta,$$

it follows immediately from the chain rule that

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = (\cos\theta)\frac{\partial u}{\partial x} + (\sin\theta)\frac{\partial u}{\partial y}.$$

Similarly, since

$$\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(r\cos\theta) = -r\sin\theta, \quad \frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta}(r\sin\theta) = r\cos\theta,$$

it follows that

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = (-r\sin\theta)\frac{\partial u}{\partial x} + (r\cos\theta)\frac{\partial u}{\partial y}.$$

It is often convenient to write the results as operator equations,

$$\frac{\partial}{\partial r} = (\cos\theta)\frac{\partial}{\partial x} + (\sin\theta)\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial \theta} = (-r\sin\theta)\frac{\partial}{\partial x} + (r\cos\theta)\frac{\partial}{\partial y}.$$

For the second derivatives, we find that

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right) &= \frac{\partial}{\partial r}\left[\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\right] \\ &= \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) + \sin\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right) \\ &= \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) &= \frac{\partial}{\partial \theta}\left[-r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}\right] \\ &= -r\sin\theta\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right) + r\cos\theta\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial y}\right) - r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y} \\ &= r^2\sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2r^2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} + r^2\cos^2\theta\frac{\partial^2 u}{\partial y^2} - r\frac{\partial u}{\partial r},\end{aligned}$$

which yields

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\frac{\partial u}{\partial r}.$$

Adding these results together, we obtain

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{1}{r}\frac{\partial u}{\partial r}=\Delta u-\frac{1}{r}\frac{\partial u}{\partial r},$$

or equivalently,

$$\Delta u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}.$$

Finally, we can write this result in the form

$$\Delta u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}. \tag{5.25}$$

This formula for the Laplace operator, together with the theory of Fourier series, allows us to solve the Dirichlet problem for Laplace's equation in a circular disk. Indeed, we can now formulate that Dirichlet problem as follows: Find $u(r,\theta)$, for $0<r\leq 1$, such that

1. u satisfies Laplace's equation,
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0, \tag{5.26}$$
2. u satisfies the periodicity condition $u(r,\theta+2\pi)=u(r,\theta)$,
3. u is well-behaved near $r=0$,
4. u satisfies the boundary condition $u(1,\theta)=h(\theta)$, where $h(\theta)$ is a given well-behaved function satisfying the periodicity condition $h(\theta+2\pi)=h(\theta)$.

The first three of these conditions are homogeneous linear. To treat these conditions via the method of separation of variables, we set

$$u(r,\theta)=R(r)\Theta(\theta), \quad \text{where} \quad \Theta(\theta+2\pi)=\Theta(\theta).$$

Substitution into (5.26) yields

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)\Theta+\frac{R}{r^2}\frac{d^2\Theta}{d\theta^2}=0.$$

We multiply through by r^2,

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right)\Theta+R\frac{d^2\Theta}{d\theta^2}=0,$$

and divide by $-R\Theta$ to obtain

$$-\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}.$$

The left-hand side of this equation does not depend on θ while the right-hand side does not depend on r. Thus neither side can depend on either θ or r, and hence both sides must be constant:

$$-\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \lambda.$$

Thus the partial differential equation divides into two ordinary differential equations

$$\frac{d^2\Theta}{d\theta^2} = \lambda\Theta, \quad \Theta(\theta + 2\pi) = \Theta(\theta),$$

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\lambda R.$$

We have seen the first of these equations before when we studied heat flow in a circular wire, and we recognize that with the periodic boundary conditions, the only nontrivial solutions are

$$\lambda = 0, \quad \Theta = \frac{a_0}{2},$$

where a_0 is a constant, and

$$\lambda = -n^2, \quad \Theta = a_n \cos n\theta + b_n \sin n\theta,$$

where a_n and b_n are constants, for n a positive integer. Substitution into the second equation yields

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) - n^2 R = 0.$$

If $n = 0$, the equation for R becomes

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0,$$

which is easily solved to yield

$$R(r) = A + B\log r,$$

where A and B are constants of integration. In order for this equation to be well-behaved as $r \to 0$ we must have $B = 0$, and the solution $u_0(r,\theta)$ to Laplace's equation in this case is constant.

When $n \neq 0$, the equation for R is a Cauchy-Euler equidimensional equation and we can find a nontrivial solution by setting

$$R(r) = r^m.$$

Then the equation becomes

$$r\frac{d}{dr}\left(r\frac{d}{dr}(r^m)\right) = n^2 r^m,$$

and carrying out the differentiation on the left-hand side yields the characteristic equation

$$m^2 - n^2 = 0,$$

which has the solutions $m = \pm n$. In this case, the solution is

$$R(r) = Ar^n + Br^{-n}.$$

Once again, in order for this equation to be well-behaved as $r \to 0$ we must have $B = 0$, so $R(r)$ is a constant multiple of r^n, and

$$u_n(r, \theta) = a_n r^n \cos n\theta + b_n r^n \sin n\theta.$$

The general solution to (5.26) which is well-behaved at $r = 0$ and satisfies the periodicity condition $u(r, \theta + 2\pi) = u(r, \theta)$ is therefore

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n r^n \cos n\theta + b_n r^n \sin n\theta),$$

where $a_0, a_1, \ldots, b_1, \ldots$ are constants. To determine these constants we must apply the boundary condition:

$$h(\theta) = u(1, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

We conclude that the constants $a_0, a_1, \ldots, b_1, \ldots$ are simply the Fourier coefficients of h.

Exercises:

5.6.1. Solve the following boundary value problem for Laplace's equation in a disk: Find $u(r, \theta), 0 < r \leq 1$, such that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

$$u(r, \theta + 2\pi) = u(r, \theta), \qquad \text{u well-behaved near } r = 0$$

and

$$u(1, \theta) = h(\theta), \quad \text{where} \quad h(\theta) = 1 + \cos\theta - 2\sin\theta + 4\cos 2\theta.$$

5.6.2. Solve the following boundary value problem for Laplace's equation in a disk: Find $u(r, \theta), 0 < r \leq 1$, such that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

$$u(r, \theta + 2\pi) = u(r, \theta), \qquad \text{u well-behaved near } r = 0$$

and

$$u(1, \theta) = h(\theta),$$

where $h(\theta)$ is the periodic function such that

$$h(\theta) = |\theta|, \qquad \text{for } -\pi \leq \theta \leq \pi.$$

5.6.3. Solve the following boundary value problem for Laplace's equation in an annular region: Find $u(r, \theta), 1 \leq r \leq 2$, such that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \qquad u(r, \theta + 2\pi) = u(r, \theta),$$

$$u(1, \theta) = 2\cos\theta \qquad \text{and} \qquad u(2, \theta) = \frac{5}{2}\cos\theta,$$

by carrying out the following steps:

a. First find the product solutions $u(r, \theta) = R(r)\Theta(\theta)$ to the homogenous linear part of the problem:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad u(r, \theta + 2\pi) = u(r, \theta). \tag{5.27}$$

In doing this you will need to solve a sequence of second-order Cauchy-Euler equations for $R(r)$, the solutions in each case having two constants of integration.

b. Use the superposition principle to write out the general solution to (5.27).

c. Finally use Fourier analysis to find the particular solution which satisfies the two boundary conditions

$$u(1, \theta) = 2\cos\theta, \qquad \text{and} \qquad u(2, \theta) = \frac{5}{2}\cos\theta.$$

5.6.4. Solve the following boundary value problem for Laplace's equation in an annular region: Find $u(r, \theta), 1 \leq r \leq 2$, such that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \qquad u(r, \theta + 2\pi) = u(r, \theta),$$

$$u(1, \theta) = 1 + \cos\theta - \sin\theta + 2\cos 2\theta$$

and

$$u(2, \theta) = 2\cos\theta - 2\sin\theta + \frac{17}{4}\cos 2\theta.$$

5.7 Eigenvalues of the Laplace operator

We would now like to consider the heat equation for a room whose shape is given by a well-behaved but otherwise arbitrary bounded region D in the (x, y)-plane, the boundary ∂D being a well-behaved curve. We would also like to consider the wave equation for a vibrating drum in the shape of such a region D. Both cases quickly lead to the eigenvalue problem for the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Let's start with the heat equation

$$\frac{\partial u}{\partial t} = c^2 \Delta u, \tag{5.28}$$

with the Dirichlet boundary condition,

$$u(x, y, t) = 0 \quad \text{for} \quad (x, y) \in \partial D.$$

To solve this equation, we apply separation of variables as before, setting

$$u(x, y, t) = f(x, y)g(t),$$

and substitute into (5.28) to obtain

$$f(x, y)g'(t) = c^2(\Delta f)(x, y)g(t).$$

Then we divide by $c^2 f(x, y)g(t)$,

$$\frac{1}{c^2 g(t)} g'(t) = \frac{1}{f(x, y)}(\Delta f)(x, y).$$

The left-hand side of this equation does not depend on x or y while the right-hand side does not depend on t. Hence both sides equal a constant λ, and we obtain

$$\frac{1}{c^2 g(t)} g'(t) = \lambda = \frac{1}{f(x, y)}(\Delta f)(x, y).$$

This separates into

$$g'(t) = c^2 \lambda g(t) \qquad \text{and} \qquad (\Delta f)(x, y) = \lambda f(x, y)$$

in which f is subject to the boundary condition,

$$f(x, y) = 0 \quad \text{for} \quad (x, y) \in \partial D.$$

The same method can be used to treat the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u, \tag{5.29}$$

with the Dirichlet boundary condition,

$$u(x, y, t) = 0 \quad \text{for} \quad (x, y) \in \partial D.$$

This time, substitution of $u(x, y, t) = f(x, y)g(t)$ into (5.29) yields

$$f(x, y)g''(t) = c^2(\Delta f)(x, y)g(t), \qquad \text{or} \qquad \frac{1}{c^2 g(t)} g''(t) = \frac{1}{f(x, y)}(\Delta f)(x, y).$$

Once again, both sides must equal a constant λ, and we obtain

$$\frac{1}{c^2 g(t)} g''(t) = \lambda = \frac{1}{f(x, y)}(\Delta f)(x, y).$$

This separates into

$$g''(t) = \lambda g(t) \qquad \text{and} \qquad (\Delta f)(x, y) = \lambda f(x, y),$$

in which f is once again subject to the boundary condition,

$$f(x, y) = 0 \quad \text{for} \quad (x, y) \in \partial D.$$

In both cases, we must solve the "eigenvalue problem" for the Laplace operator with Dirichlet boundary conditions. If λ is a real number, let

$$W_\lambda = \{\text{smooth functions } f(x, y) : \Delta f = \lambda f, \ f|\partial D = 0\}.$$

Here the notation $f|\partial D = 0$ means that $f(x, y) = 0$ for $(x, y) \in \partial D$. We say that λ is a *(Dirichlet) eigenvalue* of the Laplace operator Δ on D if $W_\lambda \neq 0$. Nonzero elements of W_λ are called *eigenfunctions* and W_λ itself is called the *eigenspace* for eigenvalue λ. The dimension of W_λ is called the *multiplicity* of the eigenvalue λ. The *eigenvalue problem* consist of finding the eigenvalues λ, and a basis for each nonzero eigenspace.

Once we have solved the eigenvalue problem for a given region D in the (x, y)-plane, it is easy to solve the initial value problem for the heat equation or the wave equation on this region. To do so requires only that we substitute the values of λ into the equations for g, solve for g, multiply by the eigenfunctions to give the product solutions and form the general solution as an arbitrary superposition of the product solutions. In the case of the heat equation,

$$g'(t) = c^2 \lambda g(t) \Rightarrow g(t) = (\text{constant}) e^{c^2 \lambda t},$$

while in the case of the wave equation,

$$g''(t) = c^2 \lambda g(t) \Rightarrow g(t) = (\text{constant}) \sin(c\sqrt{-\lambda}(t - t_0)).$$

In the second case, the eigenvalues determine the frequencies of a vibrating drum which has the shape of D.

Theorem. *All of the eigenvalues of Δ are negative, and each eigenvalue has finite multiplicity. The eigenvalues can be arranged in a sequence $\lambda_1, \lambda_2, \ldots$*

with $\lambda_n \to -\infty$ as $n \to \infty$. Every smooth function $f : D \to \mathbb{R}$ such that $f|\partial D = 0$ can be represented as a convergent sum of eigenfunctions. Moreover, each eigenfunction possesses continuous partial derivatives of all orders at all points of $D - \partial D$, where $D - \partial D$ denotes the set of points in D which are not on the boundary.[2]

Although the theorem is reassuring, it is usually quite difficult to determine the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots$ explicitly for a given region in the plane.

There are two important cases in which the eigenvalues can be explicitly determined by the method of separation of variables. The first is the case where D is the rectangle,

$$D = [0, a] \times [0, b] = \{(x, y) : 0 \le x \le a, 0 \le y \le b\}.$$

We saw how to solve the eigenvalue problem

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \lambda f, \qquad f(x, y) = 0 \quad \text{for} \quad (x, y) \in \partial D,$$

when we discussed the heat and wave equations for a rectangular region. The nontrivial solutions are

$$\lambda_{mn} = -\left(\frac{\pi m}{a}\right)^2 - \left(\frac{\pi n}{b}\right)^2, \quad f_{mn}(x, y) = \sin\left(\frac{\pi m x}{a}\right) \sin\left(\frac{\pi n y}{b}\right),$$

where m and n are positive integers.

For most choices of a and b each Dirichlet eigenvalue will have multiplicity one. But in the special case of the square ($a = b$) some of the Dirichlet eigenvalues can have multiplicity greater than one. For example, if $D = [0, \pi] \times [0, \pi]$, then $2^2 + 1^2 = 1^2 + 2^2 = 5$, so the eigenspace

$$W_{-5} = \{\text{smooth functions } f(x, y) : \Delta f = -5f, \ f|\partial D = 0\}$$

has dimension two. In this case, the eigenvalue -5 has multiplicity two.

A second case in which the eigenvalue problem can be solved explicitly is that of the disk,

$$D = \{(x, y) : x^2 + y^2 \le a^2\},$$

where a is a positive number. In this case, one writes the Laplace operator in polar coordinates as in §5.6, and applies separation of variables to the resulting Helmholtz equation. The solution leads to Bessel functions, as we will see in the next section. Once we have solved the eigenvalue problem it will be easy to derive the general solution to heat flow for a circular disk, or for the motion of a vibrating circular membrane.

A few more cases were worked out in the nineteenth century. Émile Matthieu was able to use elliptic coordinates to derive the eigenvalues and eigenfunctions

[2]Note the similarity between the statement of this theorem and the statement of the theorem presented in Section 4.7. In fact, the techniques used to prove the two theorems are also quite similar.

for an ellipse, and this led to the development of Matthieu's equation.[3] Gabriel Lamé was able to work out the eigenvalues and eigenfunctions for the region D bounded by an equilateral triangle..[4]

Exercises:

5.7.1. Conosider the Dirichlet eigenvalues for the square region $D = [0, \pi] \times [0, \pi]$. What is the multiplicity of the eigenvalue -8? -10? -13?

5.7.2. Show that among all rectangular vibrating membranes of area one, the square has the lowest fundamental frequency of vibration by minimizing the function

$$f(a, b) = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$

subject to the constraints $ab = 1$, $a, b > 0$. Hint: One can eliminate b by setting $a = 1/b$ and then find the minimum of the function

$$g(a) = \left(\frac{\pi}{a}\right)^2 + (\pi a)^2,$$

when $a > 0$.

5.7.3. Let D be a finite region in the (x, y)-plane bounded by a smooth curve ∂D. Suppose that the eigenvalues for the Laplace operator Δ with Dirichlet boundary conditions on D are λ_1, λ_2, ..., where

$$0 > \lambda_1 > \lambda_2 > \cdots,$$

each eigenvalue having multiplicity one. Suppose that $\phi_n(x, y)$ is a nonzero eigenfunction for eigenvalue λ_n. Show that the general solution to the heat equation

$$\frac{\partial u}{\partial t} = \Delta u$$

with Dirichlet boundary conditions (u vanishes on ∂D) is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \phi_n(x, y) e^{\lambda_n t},$$

where the b_n's are arbitrary constants.

5.7.4. Let D be a finite region in the (x, y)-plane as in the preceding problem. Suppose that the eigenvalues for the Laplace operator Δ on D with Dirichlet boundary conditions are λ_1, λ_2, ..., once again. Show that the general solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u,$$

[3] See Courant and Hilbert, *Methods of mathematical physics I*, New York, Interscience, 1953. See Chapter V, §16.3.

[4] A modern treatment of Lamé's theory is presented in Brian J. McCartin, *Eigenstructure of the equilateral triangle*, SIAM review **45** (2003), 167-187.

together with the condition that u vanishes on ∂D and the initial condition

$$\frac{\partial u}{\partial t}(x, y, 0) = 0,$$

is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \phi_n(x, y) \cos(\sqrt{-\lambda_n} t),$$

where the b_n's are arbitrary constants.

5.7.5. Let D be a finite region in the (x, y)-plane as in the preceding problems, and suppose that the eigenvalues for the Laplace operator Δ on D with Dirichlet boundary conditions are λ_1, λ_2, ..., once again. Find the general solution to the equation

$$\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} + u = \Delta u,$$

together with the condition that u vanishes on ∂D and the initial condition

$$\frac{\partial u}{\partial t}(x, y, 0) = 0.$$

5.8 Eigenvalues for the circular disk*

To calculate the eigenvalues of the disk, it is convenient to utilize polar coordinates r, θ in terms of which the Laplace operator is

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

in terms of which the eigenvalue problem becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = \lambda f, \quad f|\partial D = 0. \tag{5.30}$$

Once again, we use separation of variables and look for product solutions of the form

$$f(r, \theta) = R(r)\Theta(\theta), \quad \text{where} \quad R(a) = 0, \quad \Theta(\theta + 2\pi) = \Theta(\theta).$$

Substitution into (5.30) yields

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)\Theta + \frac{R}{r^2}\frac{d^2\Theta}{d\theta^2} = \lambda R\Theta.$$

We multiply through by r^2,

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right)\Theta + R\frac{d^2\Theta}{d\theta^2} = \lambda r^2 R\Theta,$$

and divide by $R\Theta$ to obtain

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \lambda r^2 = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}.$$

Note that in this last equation, the left-hand side does not depend on θ while the right-hand side does not depend on r. Thus neither side can depend on either θ or r, and hence both sides must be constant:

$$-\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda r^2 = \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \mu.$$

Thus in the manner now familiar, the partial differential equation divides into two ordinary differential equations

$$\frac{d^2\Theta}{d\theta^2} = \mu\Theta, \quad \Theta(\theta + 2\pi) = \Theta(\theta),$$

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \lambda r^2 R = -\mu R, \quad R(a) = 0.$$

Once again, the only nontrivial solutions are

$$\mu = 0, \quad \Theta = \frac{a_0}{2}$$

and

$$\mu = -n^2, \quad \Theta = a_n \cos n\theta + b_n \sin n\theta,$$

for n a positive integer. Substitution into the second equation yields

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (-\lambda r^2 - n^2)R = 0.$$

Let $x = \sqrt{-\lambda}r$ so that $x^2 = -\lambda r^2$. Then

$$dx = \sqrt{-\lambda}dr \quad \Rightarrow \quad x\frac{d}{dx} = \sqrt{-\lambda}r\frac{d}{\sqrt{-\lambda}dr} = r\frac{d}{dr}$$

so if we define a function $\tilde{R}$ by $\tilde{R}(x) = R(r)$, the functions R and $\tilde{R}$ representing the same dependent variable, our differential equation becomes

$$x\frac{d}{dx}\left(x\frac{d\tilde{R}}{dx}\right) + (x^2 - n^2)\tilde{R} = 0, \tag{5.31}$$

where $\tilde{R}$ vanishes when $x = \sqrt{-\lambda}a$. Finally, we change notation, replacing $\tilde{R}$ by y, and the differential equation with boundary condition becomes

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (x^2 - n^2)y = 0, \qquad y(\sqrt{-\lambda}a) = 0. \tag{5.32}$$

We recognize the differential equation in (5.32) as Bessel's equation, which is treated in Section 2.6.

In Section 2.6, it is shown that for each choice of n, Bessel's equation has a one-dimensional space solutions which are well-behaved as x approaches zero. These solutions are constant multiples of what is called the Bessel function of the first kind $J_n(x)$. Here is an important fact regarding Bessel functions of the first kind:

Theorem. *For each nonnegative integer n, $J_n(x)$ has infinitely many positive zeros.*

Graphs of the functions $J_0(x)$ and $J_1(x)$ suggest that this theorem might well be true, but it takes some effort to prove rigorously. For completeness, we sketch the proof for the case of $J_0(x)$ at the end of the section.

The zeros of the Bessel functions are used to determine the Dirichlet eigenvalues of the Laplace operator on the disk. To see how, note first that the boundary condition,

$$y(\sqrt{-\lambda}a) = 0,$$

requires that $\sqrt{-\lambda}a$ be one of the zeros of $J_n(x)$. Let $\alpha_{n,k}$ denote the k-th positive root of the equation $J_n(x) = 0$. Then

$$\sqrt{-\lambda}a = \alpha_{n,k} \quad \Rightarrow \quad \lambda = -\frac{\alpha_{n,k}^2}{a^2},$$

and

$$R(r) = J_n(\alpha_{n,k}r/a)$$

will be a solution to (5.31) vanishing at $r = a$. Hence, in the case where $n = 0$,

$$\lambda_{0,k} = -\frac{\alpha_{0,k}^2}{a^2},$$

and

$$f_{0,k}(r,\theta) = J_0(\alpha_{0,k}r/a)$$

will be a solution to the eigenvalue problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = \lambda f, \quad f|\partial D = 0.$$

Similarly, in the case of general n,

$$\lambda_{n,k} = -\frac{\alpha_{n,k}^2}{a^2},$$

and

$$f_{n,k}(r,\theta) = J_n(\alpha_{n,k}r/a)\cos(n\theta) \quad \text{or} \quad g_{n,k} = J_n(\alpha_{n,k}r/a)\sin(n\theta),$$

for $n = 1, 2, \ldots$, will be solutions to this eigenvalue problem.

According to the Theorem presented in the previous section, if $h(r, \theta)$ is any smooth function, defined for $r \leq a$ such that $h(r, \theta)$ is well-behaved as $r \to 0$, $h(r, \theta + 2\pi) = h(r, \theta)$ and $h(a, \theta) = 0$, then f can be expressed as a superposition

$$h(r, \theta) = \sum_{k=1}^{\infty} a_{0,k} f_{0,k}(r, \theta) + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} [a_{n,k} f_{n,k}(r, \theta) + b_{n,k} g_{n,k}(r, \theta)],$$

with each of the infinite series on the right converging.

Each of the eigenfunctions corresponds to a mode of oscillation for the circular vibrating membrane, the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u.$$

Suppose, for simplicity, that $a = 1$. Then

$$\alpha_{0,1} = 2.40483 \Rightarrow \lambda_{0,1} = -5.7832,$$

$$\alpha_{0,2} = 5.52008 \Rightarrow \lambda_{0,2} = -30.471,$$

$$\alpha_{0,3} = 8.65373 \Rightarrow \lambda_{0,3} = -74.887,$$

$$\alpha_{0,4} = 11.7195 \Rightarrow \lambda_{0,3} = -139.039,$$

and so forth. If in addition, $c^2 = 1$, the mode of oscillation corresponding to the function $f_{0,1}$ will vibrate with frequency

$$\frac{\sqrt{-\lambda_{0,1}}}{2\pi} = \frac{\alpha_{0,1}}{2\pi} = .38274,$$

while the mode of oscillation corresponding to the function $f_{0,2}$ will vibrate with frequency

$$\frac{\sqrt{-\lambda_{0,2}}}{2\pi} = \frac{\alpha_{0,2}}{2\pi} = .87855.$$

Similarly,

$$\alpha_{1,1} = 3.83171 \Rightarrow \lambda_{1,1} = -14.682,$$

$$\alpha_{2,1} = 5.13562 \Rightarrow \lambda_{2,1} = -26.3746,$$

and hence the mode of oscillation corresponding to the functions $f_{1,1}$ and $g_{1,1}$ will vibrate with frequency

$$\frac{\sqrt{-\lambda_{1,1}}}{2\pi} = \frac{\alpha_{1,1}}{2\pi} = .60984,$$

while the mode of oscillation corresponding to the functions $f_{2,1}$ and $g_{2,1}$ will vibrate with frequency

$$\frac{\sqrt{-\lambda_{2,1}}}{2\pi} = \frac{\alpha_{2,1}}{2\pi} = .81736.$$

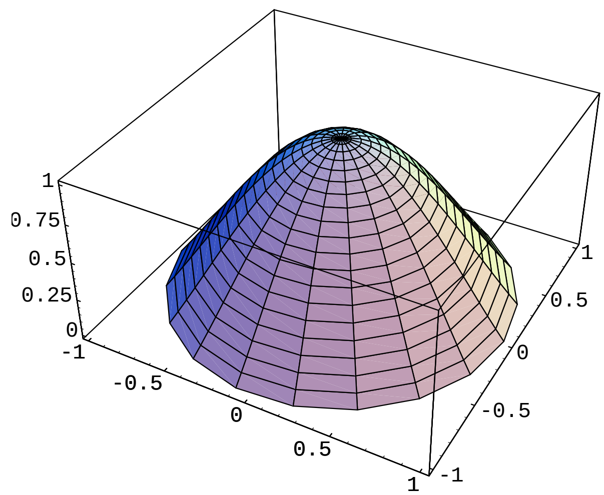

Figure 5.2: Graph of the function $f_{0,1}(r, \theta)$.

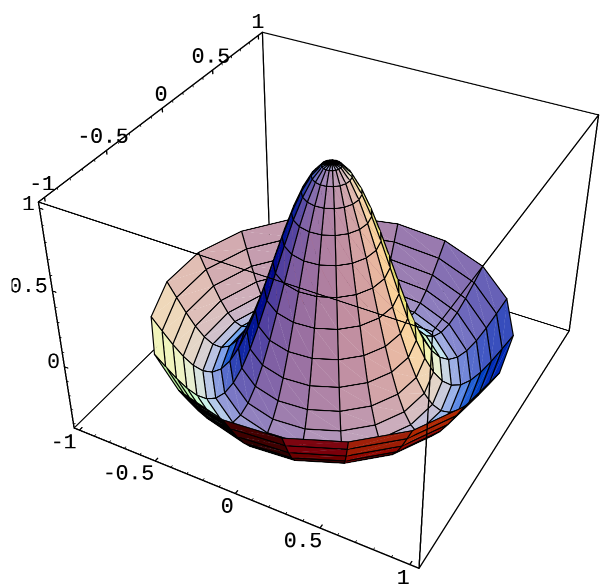

Figure 5.3: Graph of the function $f_{0,2}(r, \theta)$.

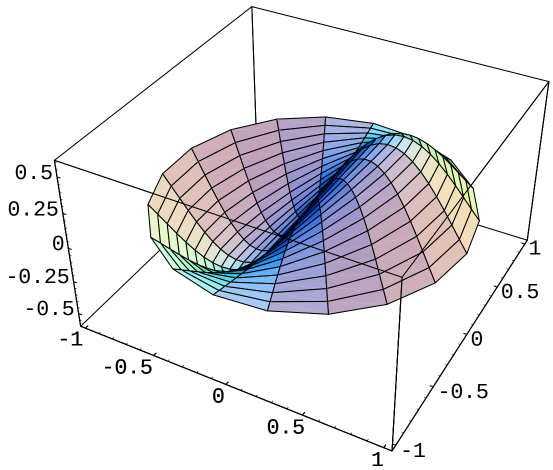

Figure 5.4: Graph of the function $f_{1,1}(r,\theta)$.

In the general case in which $c^2 = T/\rho$, these frequencies must be multiplied by c. A general vibration of the vibrating membrane will be a superposition of the modes of oscillation we have constructed. The fact that the frequencies of oscillation of a circular drum are not integral multiples of a single fundamental frequency (as in the case of the violin string) limits the extent to which a circular drum can be tuned to a specific tone.

Proof that $J_0(x)$ has infinitely many positive zeros: First, we make a change of variables $x = e^z$ and note that as z ranges over the real numbers, the corresponding variable x ranges over all the positive real numbers. Since

$$dx = e^z dz, \qquad \frac{d}{dx} = \frac{1}{e^z}\frac{d}{dz} \qquad \text{and hence} \qquad x\frac{d}{dx} = e^z\frac{1}{e^z}\frac{d}{dz} = \frac{d}{dz}.$$

Thus Bessel's equation (5.32) in the case where $n = 0$ becomes

$$\frac{d^2y}{dz^2} + e^{2z}y = 0. \tag{5.33}$$

Suppose that $z_0 > 1$ and $y(z)$ is a solution to (5.33) with $y(z_0) \neq 0$. We claim that $y(z)$ must change sign at some point between z_0 and $z_0 + \pi$.

Assume first that $y(z_0) > 0$. Let $f(z) = \sin(z - z_0)$. Since $f''(z) = -f(z)$ and $y(z)$ satisfies (5.33),

$$\frac{d}{dz}\left[y(z)f'(z) - y'(z)f(z)\right] = y(z)f''(z) - y''(z)f(z)$$

$$= -y(z)f(z) + e^{2z}y(z)f(z) = (e^{2z} - 1)y(z)f(z).$$

Note that $f(z) > 0$ for z between z_0 and $z_0 + \pi$. If also $y(z) > 0$ for z between z_0 and $z_0 + \pi$, then $y(z)f(z) > 0$ and

$$0 < \int_{z_0}^{z_0+\pi} (e^{2z} - 1)y(z)f(z)dz = [y(z)f'(z) - y'(z)f(z)]_{z_0}^{z_0+\pi}$$

$$= y(z_0 + \pi)f'(z_0 + \pi) - y(z_0)f(z_0) = -y(z_0 + \pi) - y(z_0) < 0,$$

a contradiction. Hence our assumption that $y(z)$ be postive for z between z_0 and $z_0 + \pi$ must be incorrect, $y(z)$ must change sign at some z between z_0 and $z_0 + \pi$, and hence $y(z)$ must be zero at some point in this interval.

If $y(z_0) < 0$, just apply the same argument to $-y$. In either case, we conclude that $y(z)$ must be zero at some point in any interval to the right of $z = 1$ of length at least π. It follows that the solution to (5.33) must have infinitely many zeros in the region $z > 1$, and $J_0(x)$ must have infinitely many positive zeros, as claimed.

The fact that $J_n(x)$ has infinitely many positive zeros could be proven in a similar fashion.

Exercises:

5.8.1. Solve the following initial value problem for the heat equation in a disk: Find $u(r, \theta, t), 0 < r \leq 1$, such that

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

$$u(r, \theta + 2\pi, t) = u(r, \theta, t), \quad \text{u well-behaved near } r = 0,$$

$$u(1, \theta, t) = 0,$$

$$u(r, \theta, 0) = J_0(\alpha_{0,1}r) + 3J_0(\alpha_{0,2}r) - 2J_1(\alpha_{1,1}r)\sin\theta + 4J_2(\alpha_{2,1}r)\cos 2\theta.$$

5.8.2. Solve the following initial value problem for the vibrating circular membrane: Find $u(r, \theta, t), 0 < r \leq 1$, such that

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

$$u(r, \theta + 2\pi, t) = u(r, \theta, t), \quad \text{u well-behaved near } r = 0,$$

$$u(1, \theta, t) = 0, \quad \frac{\partial u}{\partial t}(r, \theta, 0) = 0,$$

$$u(r, \theta, 0) = J_0(\alpha_{0,1}r) + 3J_0(\alpha_{0,2}r) - 2J_1(\alpha_{1,1}r)\sin\theta + 4J_2(\alpha_{2,1}r)\cos 2\theta.$$

5.8.3. (For students with access to Mathematica) a. Run the following Mathematica program to sketch the Bessel function $J_0(x)$:

```
n=0; Plot[ BesselJ[n,x], {x,0,15}]
```

b. From the graph it is clear that the first root of the equation $J_0(x) = 0$ is near 2. Run the following Mathematica program to find the first root $\alpha_{0,1}$ of the Bessel function $J_0(x)$:

```
n=0; FindRoot[ BesselJ[n,x] == 0 ,{x,2}]
```

Find the next two nonzero roots of the Bessel function $J_0(x)$.

c. Modify the programs to sketch the Bessel functions $J_1(x), \ldots, J_5(x)$, and determine the first three nonzero roots of each of these Bessel functions.

5.8.4. Which has a lower fundamental frequency of vibration, a square drum or a circular drum of the same area?

5.8.5. What happens to the fundamental frequency of a vibrating circular drum when its radius is doubled?

5.9 Fourier analysis for the circular vibrating membrane*

To finish up the solution to the initial value problem for arbitrary initial displacements of the vibrating membrane, we need to develop a generalization of the theory Fourier series which allows us to find a decomposition of an function representing an initial value into eigenfunctions which solve the eigenvalue problem for the Laplace operator. In other words, we need a theory which allows us to decompose an arbitrary perturbation of the equilibrium position for a vibrating membrane into a superposition of the various modes of oscillation.

Such a theory can be developed for an arbitrary bounded region D in the (x, y)-plane bounded by a smooth closed curve ∂D. Let V denote the space of smooth functions $f : D \to \mathbb{R}$ whose restrictions to ∂D vanish. It is important to observe that V is a "vector space": the sum of two elements of V again lies in V and the product of an element of V with a constant again lies in V.

We define an inner product $\langle , \rangle$ on V by

$$\langle f, g\rangle = \int\int_D f(x, y)g(x, y)dxdy.$$

Lemma. *With respect to this inner product, eigenfunctions corresponding to distinct eigenvalues are perpendicular; if f and g are smooth functions vanishing on ∂D such that*

$$\Delta f = \lambda f, \quad \Delta g = \mu g, \tag{5.34}$$

then either $\lambda = \mu$ or $\langle f, g\rangle = 0$.

The proof of this lemma is a nice application of Green's theorem from vector calculus. Indeed, it follows from Green's theorem that

$$\int_{\partial D} -f\frac{\partial g}{\partial y}dx + f\frac{\partial g}{\partial x}dy = \int\int_D \left(\frac{\partial}{\partial x}[f(\partial g/\partial x)] - \frac{\partial}{\partial y}[-f(\partial g/\partial y)]\right) dxdy$$

$$= \int\int_D \left[\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right] dxdy + \int\int_D f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right) dxdy.$$

Hence if the restriction of f to ∂D vanishes,

$$\int\int_D f\Delta g dxdy = -\int\int_D \nabla f \cdot \nabla g dxdy.$$

Similarly, if the restriction of g to ∂D vanishes,

$$\int\int_D g\Delta f dxdy = -\int\int_D \nabla f \cdot \nabla g dxdy.$$

Thus if f and g lie in V,

$$\langle f, \Delta g\rangle = -\int\int_D \nabla f \cdot \nabla g dxdy = \langle g, \Delta f\rangle.$$

In particular, if (5.34) holds, then

$$\mu\langle f, g\rangle = \langle f, \Delta g\rangle = \langle \Delta f, g\rangle = \lambda\langle f, g\rangle,$$

and hence

$$(\lambda - \mu)\langle f, g\rangle = 0.$$

It follows immediately that either $\lambda - \mu = 0$ or $\langle f, g\rangle = 0$, and the lemma is proven.

Now let us focus on the special case in which D is a circular disk. Recall the problem that we want to solve, in terms of polar coordinates: Find

$$u(r, \theta, t), 0 < r \leq 1,$$

so that

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

$$u(r, \theta + 2\pi, t) = u(r, \theta, t), \quad \text{u well-behaved near } r = 0,$$

$$u(1, \theta, t) = 0,$$

$$u(r, \theta, 0) = h(r, \theta), \quad \frac{\partial u}{\partial t}(r, \theta, 0) = 0.$$

It follows from the argument in the preceding section that the general solution to the homogeneous linear part of the problem can be expressed in terms of the eigenfunctions

$$f_{n,k}(r, \theta) = J_n(\alpha_{n,k} r/a)\cos(n\theta), \quad g_{n,k} = J_n(\alpha_{n,k} r/a)\sin(n\theta).$$

Indeed, the general solution must be a superposition of the products of these eigenfunctions with periodic functions of t of frequencies $\sqrt{-\lambda_{n,k}}/2\pi$. Thus this general solution is of the form

$$u(r,\theta,t) = \sum_{k=1}^{\infty} a_{0,k} f_{0,k}(r,\theta)\cos(\alpha_{0,k}t) + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}[a_{n,k}f_{n,k}(r,\theta) + b_{n,k}g_{n,k}(r,\theta)]\cos(\alpha_{n,k}t). \quad (5.35)$$

We need to determine the $a_{n,k}$'s and the $b_{n,k}$'s so that the inhomogeneous initial condition $u(r,\theta,0) = h(r,\theta)$ is also satisfied. If we set $t = 0$ in (5.35), this condition becomes

$$\sum_{k=1}^{\infty} a_{0,k}f_{0,k} + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}[a_{n,k}f_{n,k} + b_{n,k}g_{n,k}] = h. \quad (5.36)$$

Thus we need to express an arbitrary initial displacement $h(r,\theta)$ as a superposition of these eigenfunctions.

It is here that the inner product $\langle , \rangle$ on V comes to the rescue. The lemma implies that eigenfunctions corresponding to distinct eigenvalues are perpendicular. This implies, for example, that

$$\langle f_{0,j}, f_{0,k}\rangle = 0, \quad \text{unless } j = k.$$

Similarly, for arbitrary $n \geq 1$,

$$\langle f_{n,j}, f_{n,k}\rangle = \langle f_{n,j}, g_{n,k}\rangle = \langle g_{n,j}, g_{n,k}\rangle = 0, \quad \text{unless } j = k.$$

The lemma does not immediately imply that

$$\langle f_{n,j}, g_{n,j}\rangle = 0.$$

To obtain this relation, recall that in terms of polar coordinates, the inner product $\langle , \rangle$ on V takes the form

$$\langle f, g\rangle = \int_0^1 \left[\int_0^{2\pi} f(r,\theta)g(r,\theta)rd\theta\right] dr.$$

If we perform integration with respect to θ first, we can conclude from the familiar integral formula

$$\int_0^{2\pi} \sin n\theta \cos n\theta d\theta = 0$$

that

$$\langle f_{n,j}, g_{n,j}\rangle = 0,$$

as desired.

Moreover, using the integral formulae

$$\int_0^{2\pi} \cos n\theta \cos m\theta d\theta = \begin{cases} \pi, & \text{for } m = n, \\ 0, & \text{for } m \neq n, \end{cases}$$

$$\int_0^{2\pi} \sin n\theta \sin m\theta d\theta = \begin{cases} \pi, & \text{for } m = n, \\ 0, & \text{for } m \neq n, \end{cases}$$

$$\int_0^{2\pi} \sin n\theta \cos m\theta d\theta = 0,$$

we can check that

$$\langle f_{n,j}, f_{m,k} \rangle = \langle g_{n,j}, g_{m,k} \rangle = \langle f_{n,j}, g_{m,k} \rangle = 0,$$

unless $m = n$. It then follows from the lemma that

$$\langle f_{n,j}, f_{m,k} \rangle = \langle g_{n,j}, g_{m,k} \rangle = \langle f_{n,j}, g_{m,k} \rangle = 0,$$

unless $j = k$ and $m = n$.

From these relations, it is not difficult to construct formulae for the coefficients of the generalized Fourier series of a given function $h(r, \theta)$. Indeed, if we take the inner product of equation (5.36) with $u_{n,k}$, we obtain

$$a_{n,k} \langle f_{n,k}, f_{n,k} \rangle = \langle h, f_{n,k} \rangle,$$

or equivalently

$$a_{n,k} = \frac{\langle h, f_{n,k} \rangle}{\langle f_{n,k}, f_{n,k} \rangle}. \tag{5.37}$$

Similarly,

$$b_{n,k} = \frac{\langle h, g_{n,k} \rangle}{\langle g_{n,k}, g_{n,k} \rangle}. \tag{5.38}$$

Thus we finally see that the solution to our initial value problem is

$$u(r, \theta, t) = \sum_{k=1}^{\infty} a_{0,k} f_{0,k}(r, \theta) \cos(\alpha_{0,k} t)$$
$$+ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} [a_{n,k} f_{n,k}(r, \theta) + b_{n,k} g_{n,k}(r, \theta)] \cos(\alpha_{n,k} t),$$

where the coefficients are determined by the integral formulae (5.37) and (5.38).

Exercises:

5.9.1. (For students with access to Mathematica) Suppose that a circular vibrating drum is given the initial displacement from equilibrium described by the function

$$h(r, \theta) = .1(1 - r),$$

for $0 \leq r \leq 1$. In order to find the solution to the wave equation with this initial condition, we need to expand $h(r,\theta)$ in terms of eigenfunctions of Δ. Because of axial symmetry, we can write

$$h(r,\theta) = a_{0,1}J_0(\alpha_{0,1}r) + a_{0,2}J_0(\alpha_{0,2}r) + \dots.$$

a. Use the following Mathematica program to determine the coefficient $a_{0,1}$ in this expansion:

```
d = NIntegrate[r (BesselJ[0,2.40483 r])∧2,{r,0,1}];
n = NIntegrate[r (BesselJ[0,2.40483 r]) .1 (1-r),{r,0,1}];
a01 = n/d
```

b. Use a modification of the program to determine the coefficients $a_{0,2}$, $a_{0,3}$, and $a_{0,4}$.

c. Determine the first four terms in the solution to the initial value problem: Find

$$u(r,\theta,t), 0 < r \leq 1,$$

so that

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

$$u(r,\theta+2\pi,t) = u(r,\theta,t), \quad \text{u well-behaved near } r=0,$$

$$u(1,\theta,t) = 0,$$

$$u(r,\theta,0) = .1(1-r), \quad \frac{\partial u}{\partial t}(r,\theta,0) = 0.$$

5.9.2. (For students with access to Mathematica) The following program will sketch graphs of the successive positions of a membrane, vibrating with a superposition of several of the lowest modes of vibration:

```
a01 = .2; a02 = .1; a03 = .2; a04 = .1; a11 = .3; b11 = .2;
vibmem = Table[
    ParametricPlot3D[ { r Cos[u], r Sin[u] ,
        a01 BesselJ[0,2.4 r] Cos[2.4 t]
        + a02 BesselJ[0,5.52 r] Cos[5.52 t]
        + a03 BesselJ[0,8.65 r] Cos[8.65 t]
        + a04 BesselJ[0,11.8 r] Cos[11.8 t]
        + a11 BesselJ[1,3.83 r] Cos[u] Cos[3.83 t]
        + b11 BesselJ[1,3.83 r] Sin[u] Cos[3.83 t] },
        {r,0,1}, {u,0,2 Pi}, PlotRange -> {-.5,.5}
    ], {t,0,1.,.1}
]
```

a. Execute this program and then animate the sequence of graphics by double-clicking on the first cell in the table.

b. Replace the values of $a01, \ldots$ with the Fourier coefficients obtained in the preceding problem, execute the resulting program to generate a sequence of graphics, and animate the sequence as before.

5.10 Heat in three dimensions

Of course, all of the preceding theory can be extended to the Laplace operator in three dimensions, and this extension is quite important.

One of the many applications of the Laplace operator in three dimensions is to determine the steady-state distribution of temperature in a region D within (x, y, z)-space, when the temperature on the boundary ∂D is given. This is called the *Dirichlet problem* for the Laplace operator. In more advanced books the following theorem is proven:

Theorem. *Let D be a bounded region in the (x, y, z)-space which is bounded by a sufficiently well-behaved surface ∂D, and let $\phi : \partial D \to \mathbb{R}$ be a continuous function. Then there is a unique function $u : D \to \mathbb{R}$ such that*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{for } (x, y, z) \in D \text{ and}$$

$$u(x, y, z) = \phi(x, y, z), \qquad \text{for } (x, y, z) \in \partial D.$$

Moreover, u has continuous partial derivatives of arbitrary orders at every point (x, y, z) which lies in D but not on ∂D.

For regions in (x, y, z)-space that are sufficiently simple, a rectangular solid, a right circular cylinder or a ball bounded by a sphere, an explicit solution to Dirichlet's problem can be found by the techniques of separation of variables and Fourier analysis presented on the previous pages.

For studying these regions, there are three frequently used coordinate systems, the rectangular cartesion coordinates (x, y, z), in terms of which the Laplace operator is of course

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{5.39}$$

cylindrical coordinates (r, θ, z),

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

in terms of which the Laplace operator is

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}, \tag{5.40}$$

and spherical coordinates. Spherical coordinates (ρ, ϕ, θ) are defined by

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi,$$

or in terms of cylindrical coordinates by

$$r = \rho \sin\phi, \quad \theta = \theta, \quad z = \rho \cos\phi,$$

where $-\pi/2 < \phi < \pi/2$. Using the transformation $(r = \rho \sin\phi, z = \rho \cos\phi)$ and the chain rule, we can determine the Laplace operator in spherical coordinates. We will leave the derivation to the diligent reader. The result is

$$\Delta = \frac{1}{\rho^2 \sin\phi}\left[\frac{\partial}{\partial\rho}\left(\rho^2 \sin\phi \frac{\partial}{\partial\rho}\right) + \frac{\partial}{\partial\phi}\left(\sin\phi \frac{\partial}{\partial\phi}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right)\right]. \quad (5.41)$$

Using (5.39), (5.40) and (5.41), we can solve not only the Dirichlet problem, but many other three-dimensional problems involving Laplace's equation, such as the heat equation and the wave equation.

In this section, we focus first on the Dirichlet problem for the Laplace operator Δ in a ball. Thus suppose we specify the temperature on the boundary ∂B of a spherical ball

$$B = \{(x, y, z) \in R^3 : x^2 + y^2 + z^2 \leq 1\}$$

and want to determine the equilibrium temperature on the inside. Clearly spherical coordinates are the ones to use for this problem.

More precisely, we seek a function $u(\rho, \theta, \phi)$, for $0 \leq \rho \leq 1$ and $0 \leq \phi \leq \pi$, which satisfies the following conditions:

1. $\Delta u = 0$, where Δ is given by (5.41),

2. $u(\rho, \theta + 2\pi, \phi) = u(\rho, \theta, \phi)$,

3. $u(\rho, \theta, \phi) = u(\rho, \theta, \phi)$ well-behaved when ρ approaches 0 or ϕ approaches either 0 or π.

4. $u(1, \theta, \phi) = h(\theta, \phi)$, where h is a given function on ∂B, smooth and well-behaved near the north and south poles.

To solve this problem, we substitute $u(\rho, \theta, \phi) = R(\rho) f(\theta, \phi)$ into the equation

$$\frac{\partial}{\partial\rho}\left(\rho^2 \sin\phi \frac{\partial u}{\partial\rho}\right) + \frac{\partial}{\partial\phi}\left(\sin\phi \frac{\partial u}{\partial\phi}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial}{\partial u\theta}\right) = 0$$

and obtain

$$\left[\frac{\partial}{\partial\phi}\left(\sin\phi \frac{\partial f}{\partial\phi}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial f}{\partial\theta}\right)\right] R = -\sin\phi \frac{d}{d\rho}\left(\rho^2 \frac{dR}{d\rho}\right).$$

Dividing by $Rf \sin\phi$ yields

$$\frac{\frac{1}{\sin\phi}\left[\frac{\partial}{\partial\phi}\left(\sin\phi \frac{\partial f}{\partial\phi}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial f}{\partial\theta}\right)\right]}{f(\phi, \theta)} = -\frac{(d/d\rho)(\rho^2 R'(\rho))}{R(\rho)}.$$

We set both sides equal to a separating constant λ and obtain the partial differential equation

$$Lf = \lambda f, \quad \text{where} \quad L = \frac{1}{\sin\phi}\left[\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right)\right] \tag{5.42}$$

$$\text{and} \qquad \frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) = -\lambda R.$$

In more formal notation, we let

$$V = \{\text{ smooth functions } h : \partial B \to R\},$$

and let

$$W_\lambda = \{f \in V : Lf = \lambda f\},$$

L being defined by (5.42). As before, we say that λ is an eigenvalue if $W_\lambda \neq 0$. Moreover, if λ is an eigenvalue, W_λ is its eigenspace and nonzero elements of W_λ are called eigenfunctions.

As we will describe shortly, it can be shown that the eigenvalues are $-n(n+1)$, for $n = 0, 1, 2, \ldots$ and

$$\dim W_{-n(n+1)} = 2n + 1.$$

Thus the eigenspaces have larger and larger dimension as $n \to \infty$.

I. Axial Symmetry. We begin by analyzing the case the axially symmetric case, the case that f does not depend on θ. In this case, the operator L simplifies considerably:

$$L = \frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{d}{d\phi}\right),$$

and the eigenvalue equation is just

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{df}{d\phi}\right) = \lambda f. \tag{5.43}$$

If we make the substitution $z = \cos\phi$, we find that

$$dz = -\sin\phi d\phi, \qquad \frac{d}{d\phi} = -\sin\phi\frac{d}{dz}.$$

Thus if we define $\tilde{f}(z) = \tilde{f}(\cos\phi) = f(\phi)$, then (5.43) becomes

$$\frac{d}{dz}\left((1-z^2)\frac{d\tilde{f}}{dz}\right) = \lambda\tilde{f}$$

or

$$(1-z^2)\frac{d^2\tilde{f}}{dz^2} - 2z\frac{d\tilde{f}}{dz} - \lambda\tilde{f} = 0,$$

which we recognize as Legendre's differential equation that we studied earlier.

We need solutions which are well-behaved when $z = \pm 1$. But we saw in § 1.2 that the only solutions to Legendre's equation which are well-behaved for all z such that $-1 \leq z \leq 1$ are polynomials, and these only occur when $\lambda = -n(n+1)$, for $n = 0, 1, 2, \ldots$. Appropriately normalized, the polynomial solutions are known as the *Legendre polynomials* and are denoted by $P_n(z)$. The first few Legendre polynomials are

$$P_0(z) = 1, \quad P_1(z) = z, \quad P_2(z) = \frac{1}{2}(3z^2 - 1),$$
$$P_3(z) = \frac{1}{2}(5z^3 - 3z), \quad P_4(z) = \frac{1}{8}(35z^4 - 30z^2 = 3).$$

Any constant multiple of the Legendre polynomial $P_n(z)$ is a solution to the differential equation

$$(1 - z^2)\frac{d^2\tilde{f}}{dz^2} - 2z\frac{d\tilde{f}}{dz} + n(n+1)\tilde{f} = 0.$$

Thus, recalling that $z = \cos\phi$, we see that the only well-behaved solutions to (5.43) are

$$\lambda = -n(n+1), \quad f(\phi) = (\text{constant})P_n(\cos\phi),$$

for $n = 0, 1, 2, \ldots$.

The corresponding equation for $R(\rho)$ is

$$\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) = n(n+1)R,$$

which is a Cauchy-Euler equation. Its solutions are found to be

$$n = 0: \quad R = a + b\log|\rho|,$$

$$n > 0: \quad R = a\rho^n + \frac{b}{\rho^{n+1}},$$

where a and b are arbitrary constants. The only solutions well-behaved near $\rho = 0$ are $R = a$ when $n = 0$ and $R = a\rho^n$ when $n > 0$, and the corresponding solutions $u(r, \phi)$ to Laplace's equation are constant multiples of

$$u_0(\rho, \phi) = 1, \qquad u_n(\rho, \phi) = \rho^n P_n(\cos\phi). \tag{5.44}$$

Let us return to the problem considered before, with u independent of θ. We seek a function $u(\rho, \phi)$, for $0 \leq \rho \leq 1$ and $0 \leq \phi \leq \pi$, which satisfies the following conditions:

1. $\Delta u = 0$,

2. $u(\rho, \phi)$ is well-behaved when ρ approaches 0 and when ϕ approaches either 0 or π.

3. $u(1, \phi) = h(\phi)$, where h is a given function on ∂B, invariant under rotations around the z-axis, which is smooth and well-behaved near the north and south poles.

The general solution to the homogeneous linear part of this problem is a superposition of the product solutions (5.44).

$$u(\rho, \phi) = \sum_{n=0}^{\infty} a_n \rho^n P_n(\cos\phi).$$

To find the solution which satisfies the nonhomogeneous condition, we must solve

$$u(1, \phi) = \sum_{n=0}^{\infty} a_n P_n(\cos\phi) = h(\phi).$$

In other words, we must expand $h(\phi)$ as a superposition of Legendre polynomials.

II. General Case. We now relax the assumption that the function f is axially symmetric. This requiring analyzing the equation (5.42):

$$Lf = \lambda f, \quad \text{where} \quad L = \frac{1}{\sin\phi}\left[\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right)\right].$$

Once again, we use separation of variables, and write

$$f(\phi, \theta) = \Phi(\phi)\Theta(\theta),$$

which puts the preceding equation in the form

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\Phi'(\phi)\right)\Theta(\theta) + \frac{1}{\sin^2\phi}\Phi(\phi)\Theta''(\theta) = \lambda\Phi(\phi)\Theta(\theta).$$

Dividing by $\Phi(\phi)\Theta(\theta)/\sin^2\phi$ yields

$$\frac{(\sin\phi)(d/d\phi)(\sin\phi\Phi'(\phi))}{\Phi(\phi)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda(\sin^2\phi),$$

which by the familiar process yields

$$\frac{(\sin\phi)(d/d\phi)(\sin\phi\Phi'(\phi))}{\Phi(\phi)} - \lambda(\sin^2\phi) = -\frac{\Theta''(\theta)}{\Theta(\theta)} = -\mu,$$

where μ is a separating constant.

Thus we find a familiar eigenvalue problem once again,

$$\Theta''(\theta) = \mu\Theta(\theta), \qquad \Theta(\theta + 2\pi) = \Theta(\theta),$$

together with an ordinary differential equation for ϕ;

$$(\sin\phi)(d/d\phi)(\sin\phi\Phi'(\phi)) - (\lambda(\sin^2\phi) - \mu)\Phi(\phi) = 0. \tag{5.45}$$

The eigenvalue problem has the familiar solution:

$$\mu = 0 \quad \text{and} \quad \Theta(\theta) = \frac{a_0}{2},$$

or

$$\mu = -m^2, \quad m \text{ a positive integer and} \quad \Theta(\theta) = a_m \cos(m\theta) + b_m \sin(m\theta).$$

As in the axially symmetric case, we substitute $z = \cos\phi$ in (5.45) and let $\tilde{\Phi}(z) = \tilde{\Phi}(\cos\phi) = \Phi(\phi)$, to obtain

$$(1 - z^2)\frac{d^2\tilde{\Phi}}{dz^2} - 2z\frac{d\tilde{\Phi}}{dz} + \left[-\lambda - \frac{m^2}{1 - z^2}\right]\tilde{\Phi} = 0, \tag{5.46}$$

a differential equation called an *associated Legendre equation*. This equation has regular singular points at $z = \pm 1$ just like Legendre's equation. We can again look for solutions to the associated Legendre differential equation which are well-behaved at $z = \pm 1$.

A direct calculation shows that if $g(z)$ is any solution to Legendre's differential equation

$$(1 - z^2)\frac{d^2 g}{dz^2} - 2z\frac{dg}{dz} - \lambda g = 0,$$

then

$$\tilde{\Phi}(z) = (1 - z^2)^{m/2}\frac{d^m g}{dz^m}(z)$$

is a solution to the associated Legendre differential equation (5.46). Just as in the case of the usual Legendre equation, it can be shown that the only solutions to (5.46) which are well-behaved at $z = \pm 1$ are those that arise from this formula when $g(z)$ is a polynomial.[5] These are the constant multiples of the so-called *associated Legendre functions*

$$P_n^m(z) = (1 - z^2)^{m/2}\frac{d^m}{dz^m}P_n(z),$$

the $P_n(z)$'s being the Legendre polynomials described before. The first few associated Legendre functions with $m > 0$ are

$$P_1^1(z) = -\sqrt{1 - z^2}, \quad P_2^1(z) = -3z\sqrt{1 - z^2}, \quad P_2^2(z) = 3(1 - z^2).$$

We can now write down the solutions to the eigenvalue problem for the operator $L : V \to V$ defined by (5.42) when $|\lambda|$ is small. A basis for the eigenspace W_0 for $\lambda = 0$ consists of $\{P_0(\cos\phi) = 1\}$. A basis for the next eigenspace W_{-2} is

$$\{P_1^0(\cos\phi), P_1^1(\cos\phi)\cos\theta, P_1^1(\cos\phi)\sin\theta\},$$

[5] Further discussion of this can be found in Courant and Hilbert, *Methods of mathematical physics*, Volume 1, Chapter 5, § 10.

a space of dimension three. A basis for the next eigenspace W_{-6} is

$$\{P_2^0(\cos\phi), P_2^1(\cos\phi)\cos\theta, P_2^1(\cos\phi)\sin\theta, P_2^2(\cos\phi)\cos 2\theta, P_2^2(\cos\phi)\sin 2\theta\},$$

a space of dimension five. A basis for $W_{-n(n+1)}$ is

$$\{P_n(\cos\phi)\} \cup \{P_n^m(\cos\phi)\cos m\theta, P_n^m(\cos\phi)\sin m\theta : 1 \le m \le n\},$$

a space of dimension $2n+1$. Elements of $W_{-n(n+1)}$ are called *spherical harmonics* of degree n, and play a major role in understanding the Laplace operator in spherical coordinates.

With the solution of the eigenvalue problem for L available, we have finished the first step in solving the Dirichlet problem for steady-state temperature in a ball for arbitrary boundary data. The second step would consist of writing an arbitrary smooth function $h(\phi,\theta)$ in V as a superposition of eigenfunctions. As might be expected, this requires the development of a new theory of generalized Fourier series, similar to that employed in the circular vibrating membrane problem.

Flow of heat in a thin spherical shell. Suppose now that D is the region between two concentric spheres:

$$D = \{(x,y,z) \in R^3 : 1 \le \rho \le 1+\epsilon\},$$

ρ being the radial spherical coordinate, where ϵ is a very small positive number. We imagine that we are given an initial distribution of heat $h(\phi,\theta)$ over the spherical shell which does not depend on the radial coordinate ρ. We would like to solve the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \\ &= c^2\frac{1}{\rho^2\sin\phi}\left[\frac{\partial}{\partial\rho}\left(\rho^2\sin\phi\frac{\partial u}{\partial\rho}\right) + \frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial u}{\partial\theta}\right)\right], \end{aligned}$$

for the temperature function $u(\rho,\phi,\theta,t)$ subject to the boundary condition

$$\frac{\partial u}{\partial\rho} = 0 \quad \text{on the boundary of } D,$$

and the initial condition

$$u(\rho,\phi,\theta,0) = h(\phi,\theta).$$

It is natural to expect that as $\epsilon \to 0$ the solution $u(\rho,\phi,\theta,t)$ becomes independent of ρ for all time t, allowing us to simplify the Laplace operator (5.41) to

$$L = \frac{1}{\sin\phi}\left[\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right)\right]. \tag{5.47}$$

The operator L is known as the *Laplace operator on the unit sphere*. When we send ϵ to zero, we can imagine heat flowing over the surface of the unit sphere.

As we have done many times, we can separate variables, setting

$$u(\rho, \phi, \theta, t) = f(\phi, \theta)g(t).$$

Substituting into the heat equation yields

$$f(\phi, \theta)g'(t) = c^2 L(f)(\phi, \theta)g(t) \quad \text{or} \quad \frac{g'(t)}{c^2 g(t)} = \frac{L(f)(\phi, \theta)}{f(\phi, \theta)},$$

which separates into two equations

$$L(f)(\phi, \theta) = \lambda f(\phi, \theta), \qquad g'(t) = c^2 g(t).$$

This leads once again to the eigenvalue problem for the operator $L : V \to V$ that we considered before. As we have seen, the eigenvalues for L are $-n(n+1)$, for $n = 0, 1, 2, \ldots$ and the corresponding eigenspaces are

$$W_{-n(n+1)} = \{P_n(\cos\phi)\} \cup \{P_n^m(\cos\phi)\cos m\theta, P_n^m(\cos\phi)\sin m\theta : 1 \le m \le n\},$$

spaces of dimension $2n + 1$. We change notation slightly and let

$$W_{-n(n+1)} = \{\phi_{n,i}(\phi, \theta) : 1 \le i \le 2n + 1\}.$$

Then the general solution to the heat equation on the sphere is found to be

$$u(\phi, \theta, t) = \sum_{n=0}^{\infty} \sum_{i=1}^{2n+1} a_{n,i}\phi_{n,i}(\phi, \theta)e^{-n^2 t}.$$

To satisfy the initial condition, we must have

$$h(\phi, \theta) = u(\phi, \theta, 0) = \sum_{n=0}^{\infty} \sum_{i=1}^{2n+1} a_{n,i}\phi_{n,i}(\phi, \theta).$$

Thus we must expand $h(\phi, \theta)$ in terms of the associated Legendre polynomials as before. Just as in the case of the circular vibrating membrane, to find the coefficients $a_{n,i}$ in complete generality, and thereby determine the particular solution satisfying the initial condition, would require the development of a generalized theory of Fourier series for Legendre functions.

Exercises:

5.10.1. Solve the following Dirichlet problem for the equilibrium heat in a spherical ball: Find a function $u(\rho, \phi)$, for $0 \le \rho \le 1$ and $0 \le \phi \le \pi$, which satisfies the following conditions:

1. $\Delta u = 0$,

2. $u(\rho, \phi)$ is well-behaved when ρ approaches 0 and when ϕ approaches either 0 or π,

3. $u(1, \phi) = 5(\cos \phi)^3 - 3 \cos \phi$.

5.10.2. Solve the following Dirichlet problem for the equilibrium heat heat in a spherical ball: Find a function $u(\rho, \phi)$, for $0 \leq \rho \leq 1$ and $0 \leq \phi \leq \pi$, which satisfies the following conditions:

1. $\Delta u = 0$,

2. $u(\rho, \phi)$ is well-behaved when ρ approaches 0 and when ϕ approaches either 0 or π,

3. $u(1, \phi) = 5(\cos \phi)^3 + 2(\cos \phi)^2 - 4 \cos \phi + 5$.

5.10.3. Solve the following Dirichlet problem for the equilibrium heat in a spherical ball: Find a function $u(\rho, \phi)$, for $0 \leq \rho \leq 1$ and $0 \leq \phi \leq \pi$, which satisfies the following conditions:

1. $\Delta u = 0$,

2. $u(\rho, \phi, \theta + 2\pi) = u(\rho, \phi, \theta)$,

3. $u(\rho, \phi, \theta)$ is well-behaved when ρ approaches 0 and when ϕ approaches either 0 or π,

4. $u(1, \phi, \theta) = (\cos \phi)^2 \cos 2\theta + 5(\cos \phi)^3 + (\cos \phi)^2 - 4 \cos \phi + 5$.

5.10.4. Solve the following initial-value problem for heat flow on the surface of a sphere of unit radius: Find a function $u(\rho, \phi, t)$, for $0 \leq \rho \leq 1$, $0 \leq \phi \leq \pi$ and $t \geq 0$, which satisfies the following conditions:

1. $(\partial u / \partial t) = Lu$, where L is defined by (5.47).

2. $u(\phi, \theta + 2\pi, t) = u(\phi, \theta, t)$,

3. $u(\phi, \theta, t)$ is well-behaved when ρ approaches 0 and when ϕ approaches either 0 or π,

4. $u(\phi, \theta, 0) = (\cos \phi)^2 \cos 2\theta + 5(\cos \phi)^3 + (\cos \phi)^2 - 4 \cos \phi + 5$.

Appendix A

Using Mathematica to solve differential equations

In solving differential equations, it is sometimes necessary to do calculations which would be prohibitively difficult to do by hand. Fortunately, computers can do the calculations for us, if they are equiped with suitable software, such as Maple, Matlab, or Mathematica. This appendix is intended to give a brief introduction to the use of Mathematica for doing such calculations.

Most computers which are set up to use Mathematica contain an on-line tutorial, "Tour of Mathematica," which can be used to give a brief hands-on introduction to Mathematica. Using this tour, it may be possible for many students to learn Mathematica without referring to lengthy technical manuals. However, there is a standard reference on the use of Mathematica, which can be used to answer questions if necessary. It is *The Mathematica Book*, by Stephen Wolfram, Fourth edition, Wolfram Media and Cambridge University Press, 1999.

We give here a very brief introduction to the use of Mathematica. After launching Mathematica, you can use it as a "more powerful than usual graphing calculator." For example, if you type in

```
(11 - 5)/3
```

the computer will perform the subtraction and division, and respond with

```
Out[1] = 2
```

The notation for multiplication is $*$, so if you type in

```
2 * (7 + 4)
```

the computer will respond with

```
Out[2] = 22
```

You can also use a space instead of $*$ to denote multiplication, so if you input

```
2 8
```

the computer will respond with

```
Out[3] = 16
```

The computer can do exact arithmetic with integers or rational numbers. For example, since $\wedge$ is the symbol used for raising to a power, if you type in

```
2∧150
```

the computer will calculate the 150th power of two and give the exact result:

```
Out[4] = 1427247692705959881058285969449495136382746624
```

On the other hand, the symbol `N` tells the computer to use an approximation with a fixed number of digits of precision, so entering

```
N[2∧150]
```

will give an approximation to the exact answer, expressed in scientific notation:

Out[5] = 1.42725 10^{45}

Real numbers which are not rational, such as π, have infinite decimal expansions which cannot be stored within a computer. However, we can approximate a real number by a finite decimal expansion with a given number of digits of precision. For example, since Mathematica uses the name `E` to denote the number e, typing in

```
N[E]
```

will tell Mathematica to give a rational approximation to e to a standard number of digits of precision:

```
Out[6] = 2.71828
```

In principle, the computer can calculate rational approximations to an arbitrary number of digits of precision. Since the number π is represented by Mathematica as `Pi`, typing in

```
N[Pi,40]
```

will tell Mathematica to give a rational approximation to π to 40 digits of precision:

```
Out[7] = 3.141592653589793238462643383279502884197
```

The computer can represent not only numbers but functions. Within Mathematica, built-in functions are described by names which begin with capital

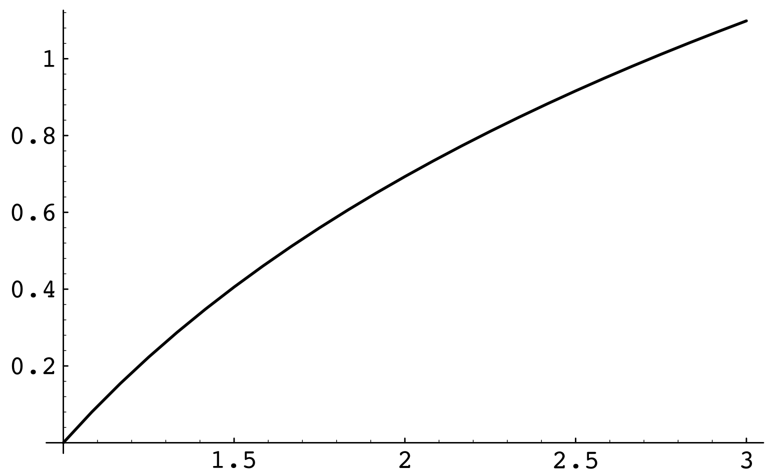

Figure A.1: Graph of the logarithm function.

letters. For example, `Log` denotes the natural base e logarithm function. Thus entering

```
N[Log[2],40]
```

will give the logarithm of 2 with 40 digits of precision:

```
Out[8] = 0.693147180559945309417232121458176568076
```

One can also plot functions with Mathematica. For example, to plot the logarithm function from the values 1 to 3, one simply inputs

```
Plot[Log[t],{t,1,3}]
```

and Mathematica will automatically produce the plot shown in Figure A.1.

We can also define functions ourselves, being careful not to capitalize them, because capitals are reserved by Mathematica for built-in functions. Thus we can define the function

$$y(t) = ce^{kt}$$

by typing

```
y[t_] := c * E∧(k * t);
```

Mathematica will remember the function we have defined until we quit Mathematica. We must remember the exact syntax of this example (use of the underline character and the colon followed by the equal sign) when defining functions. In this example c and k are *parameters* which can be defined later. Just as in the case of functions, we must be careful to represent parameters by lower case letters, so they will not be confused with built-in constants. Further entry of

```
c = 1; k = 1; N[y[1]]
```

yields the response

```
Out[11] = 2.71828
```

while entering

```
Plot[y[t],{t,0,2}]
```

will give a plot of the function we have defined, for t in the interval $[0, 2]$.

We can use Mathematica to solve matrix differential equations of the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \tag{A.1}$$

where A is a square matrix with constant entries.

The first step consists of using Mathematica to find the eigenvalues and eigenvectors of A. To see how this works, we must first become familiar with the way in which Mathematica represents matrices. Since Mathematica reserves upper case letters for descriptions of built-in functions, it is prudent to denote the matrix A by lower case a when writing statements in Mathematica. The matrix

$$A = \begin{pmatrix} -2 & 5 \\ 1 & -3 \end{pmatrix}$$

can be entered into Mathematica as a collection of row vectors,

```
a = {{-2,5},{1,-3}}
```

with the computer responding by

```
Out[1] = {{-2,5},{1,-3}}
```

Thus a matrix is thought of as a "vector of vectors." Entering

```
MatrixForm[a]
```

will cause the computer to give the matrix in the familiar form

```
Out[2] =  −2  5
          1   −3
```

To find the eigenvalues of the matrix A, we simply type

```
Eigenvalues[a]
```

and the computer will give us the exact eigenvalues

$$\frac{-5 - \sqrt{21}}{2}, \qquad \frac{-5 + \sqrt{21}}{2},$$

which have been obtained by using the quadratic formula. Quite often numerical approximations are sufficient, and these can be obtained by typing

```
eval = Eigenvalues[N[a]]
```

the response this time being

```
Out[4] = {-4.79129, -0.208712}
```

Defining `eval` to be the eigenvalues of A in this fashion, allows us to refer to the eigenvalues of A later by means of the expression `eval`.

We can also find the corresponding eigenvectors for the matrix A by typing

```
evec = Eigenvectors[N[a]]
```

and the computer responds with

```
Out[5] = {{-0.873154, 0.487445}, {0.941409, 0.337267}}
```

Putting this together with the eigenvalues gives the general solution to the original linear system (A.1) for our choice of the matrix A:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -0.873154 \\ 0.487445 \end{pmatrix} e^{-4.79129t} + c_2 \begin{pmatrix} 0.941409 \\ 0.337267 \end{pmatrix} e^{-0.208712t}.$$

Mathematica can also be used to find numerical solutions to nonlinear differential equations. The following Mathematica programs will use Mathematica's differential equation solver (which is called up by the command `NDSolve`), to find a numerical solution to the initial value problem

$$dy/dx = y(1-y), \qquad y(0) = .1,$$

give a table of values for the solution, and graph the solution curve on the interval $0 \le x \le 6$. The first step

```
sol := NDSolve[{ y'[x] == y[x] (1 - y[x]), y[0] == .1 },
y, {x,0,6}]
```

generates an "interpolation function" which approximates the solution and calls it sol, an abbreviation for solution. We can construct a table of values for the interpolation function by typing

```
Table[Evaluate[y[x] /.  sol], {x,0,6,.1}];
```

or graph the interpolation function by typing

```
Plot[Evaluate[y[x] /.  sol], {x,0,6}]
```

This leads to a plot like that shown in Figure A.2.

Readers can modify these simple programs to graph solutions to initial value problems for quite general differential equations of the canonical form

$$\frac{dy}{dx} = f(x, y).$$

All that is needed is to replace the first argument of `NDSolve` with the differential equation one wants to solve, remembering to replace the equal signs with double equal signs, as in the example.

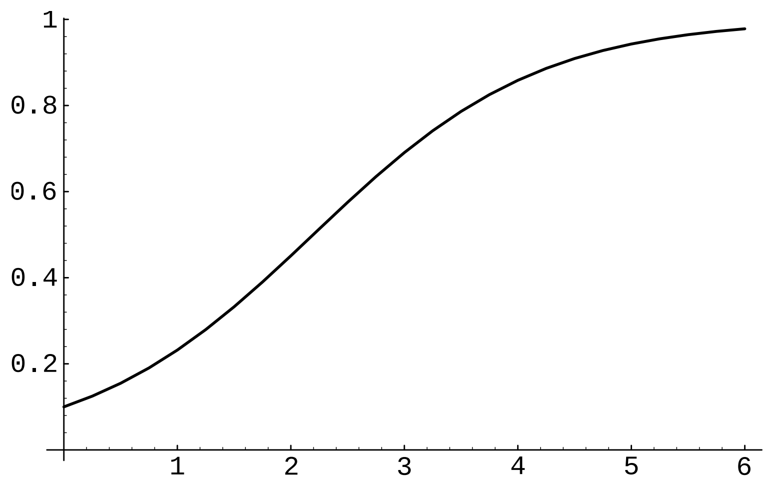

Figure A.2: Solution to $y' = y(1-y)$, $y(0) = .1$.

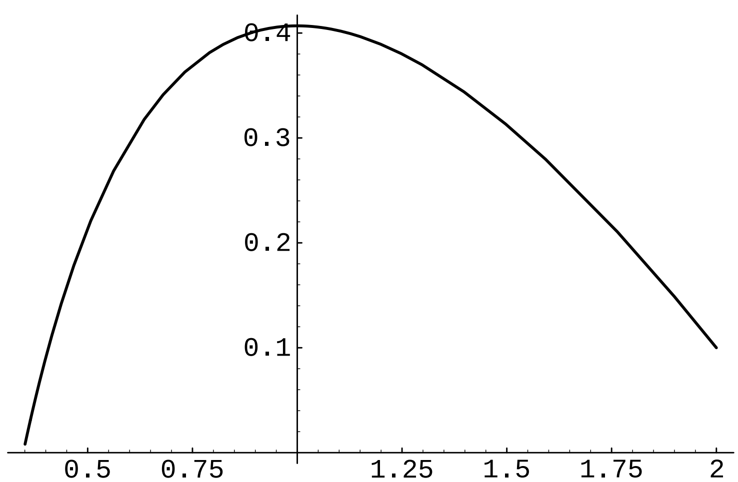

Figure A.3: A parametric plot of a solution to $dx/dt = -xy$, $dy/dt = xy - y$.

In fact, it is no more difficult to treat initial value problems for higher order equations or systems of differential equations. For example, to solve the initial value problem

$$\frac{dx}{dt} = -xy, \quad \frac{dy}{dt} = xy - y, \quad x(0) = 2, \quad y(0) = .1. \tag{A.2}$$

one simply types in

```
sol := NDSolve[{ x'[t] == - x[t] y[t], y'[t] == x[t] y[t] - y[t],
x[0] == 2, y[0] == .1 },
{x,y}, {t,0,10}]
```

Once we have this solution, we can print out a table of values for y by entering

```
Table[Evaluate[y[t] /. sol], {t,0,2,.1}]
```

We can also plot $y(t)$ as a function of t by entering

```
Plot[Evaluate[y[t] /. sol], {t,0,10}]
```

Figure A.3 shows a parametric plot of $(x(t), y(t))$ which is generated by entering

```
ParametricPlot[Evaluate[{ x[t], y[t]} /. sol], {t,0,10}]
```