Suppose that $β = (v_1, \ldots, v_n)$ is a basis for a vector space $V$. By Proposition 2.8 in the text, if $v \in V$, we can write

$$v = (v_1 \ v_2 \ldots \ v_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$  

(1)

where we are using the usual matrix multiplication on the right. We say that $(x_1, \ldots, x_n)$ are the coordinates of $v$ with respect to the basis $(v_1, \ldots, v_n)$.

For example if $V = \mathbb{R}^n$, we could take the standard basis $(e_1, \ldots, e_n)$, where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$  

Then if $x \in \mathbb{R}^n$, we can write

$$x = (e_1 \ e_2 \ldots \ e_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$  

In this case, $(x_1, \ldots, x_n)$ are the standard coordinates on $\mathbb{R}^n$.

Suppose now that $T : V \to W$ is a linear transformation, where $W$ is a vector space with basis $γ = (w_1, \ldots, w_m)$. We can then apply $T$ to equation (1) and obtain

$$T(v) = (T(v_1) \ T(v_2) \ldots \ T(v_n)) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$  

(2)

For each choice of $k$, $1 \leq k \leq n$, we $T(v_k)$ uniquely as a linear combination of $(w_1, \ldots, w_m)$:

$$T(v_k) = (w_1 \ w_2 \ldots \ w_m) \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{mk} \end{pmatrix}.$$  

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It follows that

\[
(T(v_1) \ T(v_2) \ ... \ T(v_n)) = (w_1 \ w_2 \ ... \ w_m) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},
\]

and (2) becomes

\[
T(v) = (w_1 \ ... \ w_m) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.
\] (3)

We say that

\[
A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}
\]

is the matrix of \( T \) with respect to the bases \( \beta = (v_1, \ldots, v_n) \) and \( \gamma = (w_1, \ldots, w_m) \), and write

\[ A = M(T, \beta, \gamma) = M(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m)). \]

We shorten the notation to \( A = M(T) \) when there is no danger of confusion. If

\[
T(v) = (w_1 \ ... \ w_m) \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix},
\]

then it follows from (3) that

\[
\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.
\] (4)

So the coordinates of \( T(w) \) are related to the coordinates of \( v \) by left multiplication by the matrix \( A \).

Note that if \( A \) is an \( m \times n \) matrix and \( T_A \) is the linear map from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) defined by \( T_A(x) = Ax \) it follows from (4) that with respect to the standard bases \( \mathcal{M}(T_A) = A \). The columns of \( A \) are the images of the standard basis vectors.

Suppose that we want to represent a counterclockwise rotation of \( \mathbb{R}^2 \) through an angle \( \theta \) by means of a linear map \( T \). This linear map must satisfy

\[
T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}.
\]
Thus with respect to the standard basis,
\[
(T(e_1) \ T(e_2)) = (e_1 \ e_2) \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\]
so \(M(T) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}\).

In terms of the standard coordinates on \(\mathbb{R}^2\),
\[
T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\]

Note that the basis elements transform by multiplication by \(M(T)\) on the right, while the coordinates transform by multiplication by \(M(T)\) on the left.

Here is another example. Suppose that \(P_2(\mathbb{R})\) denotes the space of polynomials of degree two, with the basis \(\beta\), which consists of
\[
p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = x^2,
\]
and that \(T : P_2(\mathbb{R}) \to P_2(\mathbb{R})\) is the linear transformation defined by
\[
T(p(x)) = p'(x) = \frac{dp}{dx}(x).
\]

Here \(V = W\) and we choose the same basis in both \(V\) and \(W\). In this case,
\[
T(1) = 0 = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad T(x) = 1 = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},
\]
\[
T(x^2) = 2x = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix},
\]
and hence
\[
M(T, \beta, \beta) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.
\]

If instead
\[
T(p(x)) = p'(x) - p(x) = \frac{dp}{dx}(x) - p(x),
\]
then
\[
T(1) = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad T(x) = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},
\]
\[
T(x^2) = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix},
\]
and hence
\[
M(T, \beta, \beta) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}.
\]