Compact symmetric bilinear forms

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joint work with:

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Sources


and numerous more recent ramifications
Main problems

Spectral analysis of the Friedrichs form

\[
[f, g] = \int_G f g dA, \quad f, g \in L^2_a(G);
\]

Asymptotics of the singular numbers of Hankel operators on multiply connected domains;

Green function estimates for certain Schrödinger operators
Tools

Hilbert space with a bilinear symmetric (compact) form $[x, y]$

complex symmetric operators $T$ w.r. to $[x, y]$

a refined polar decomposition for $T$

an abstract AAK theorem for compact $T$'s

some analysis of complex variables and potential theory
Friedrichs’ operator

$G$ bounded planar domain

d$A$ Area measure

$L^2_a(G)$ Bergman space w.r. to $dA$

$P : L^2(G) \longrightarrow L^2_a(G)$ Bergman projection

The form

$$[f, g] = \int_G f g dA = \langle f, Fg \rangle, \quad f, g \in L^2_a(G)$$

defines Friedrichs operator

$$Fg = P(\overline{g}).$$
Angle operator

$F$ is anti-linear, but $S = F^2$ is complex linear, s.a.,

$$\langle f, Sf \rangle = \|Ff\|^2,$$

and

$$Sf = PCPCf,$$

where $C$ is complex conjugation in $L^2$.

$S$ is the "angle operator" between $L^2_\alpha(G)$ and $CL^2_\alpha(G)$.

Side remark: for $h \in H^\infty(G)$, denote by $T_h$ the Toeplitz operator on $L^2_\alpha(G)$. Then

$$FT_h = T_h^* F.$$
Compactness

Assume that $G$ has real analytic, smooth boundary (not necessarily simply connected).

One can write $\bar{z} = S(z)$ with $S$ analytic in a neighborhood of $\partial G$. Thus

$$2i[f, g] = \int_G f(z)g(z)\overline{d\bar{z}} \wedge dz =$$

$$= \int_{\partial G} f(z)g(z)S(z)dz = \int fgSd\mu,$$

where $\text{supp}\mu \subset G$.

Therefore $S = F^2$ is compact.
Corners

Assume that $G$ has piece-wise smooth boundary, with finitely many corners, of interior angles $0 < \alpha_k \leq 2\pi$.

Friedrichs: For every $k$,

$$|\frac{\sin \alpha_k}{\alpha_k}|^2 \in \sigma_{ess}(S).$$

Moreover, there exists a constant $c(G) < 1$: for every $f \in L^2_\alpha(G)$,

$$\int_G f dA = 0,$$

$$|\int_G f^2 dA|^2 \leq c(G) \int_G |f|^2 dA.$$
Inverse spectral problem

P.-Shapiro: There exists a continuous family of planar domains with unitarily equivalent Friedrichs operators, and such that no two domains are related by an affine transform.

The measure $\mu$ in the example has three atoms.
Asymptotics

P.-Prokhorov: Assume $\partial G$ real analytic and smooth, and let $\lambda_k = \sqrt{(\lambda_k(S))} \geq \lambda_{k+1}$ be the eigenvalues of $F$. Then

$$\limsup_{n \to \infty} (\lambda_0 \lambda_1 \ldots \lambda_n)^{1/n^2} \leq \exp(-1/C(\partial G, \text{supp}\mu)),$$

$$\limsup_{n \to \infty} \lambda_1^{1/n} \leq \exp(-1/C(\partial G, \text{supp}\mu))$$

and

$$\liminf_{n \to \infty} \lambda_1^{1/n} \leq \exp(-2/C(\partial G, \text{supp}\mu)),$$

where $C(.,.)$ is the (Green) capacitor of the two sets.
Boundary forms

Assume that $\Gamma = \partial G$ is real analytic, smooth.

$$L^2(\Gamma) = E^2(G) \oplus E^2(G)^\perp$$

with Smirov class projections $P_\pm$.

For $a \in C(\Gamma)$ the Hankel operator

$$H_a f = P_-(af),$$

is well defined.
Approximation theory questions lead to

$$a(z) = \frac{1}{2\pi i} \int \frac{d\mu(w)}{z - w},$$

with supp\(\mu\) compact in \(G\).

Let \(R : E^2(G) \rightarrow L^2(\mu)\) be the restriction operator. Then

$$|H_a|^2 = (R^* CR)^2$$

and asymptotics (as in the planar case) can be derived with a similar proof.
**Hilbert space part in the proof**

Double orthogonal system of vectors \((u_n)\):

\[
[u_k, u_n] = \lambda_n \delta_{kn}, \quad \langle u_k, u_n \rangle = \delta_{kn},
\]

where one can choose a positive spectrum:

\[
\lambda_0 \geq \lambda_1 \geq \ldots \ldots \geq 0.
\]

**Weyl-Horn inequality:** For every system of vectors \(g_0, \ldots, g_n\) one has

\[
|\det([g_k, g_n])| \leq \lambda_0 \ldots \lambda_n \det(\langle g_k, g_n \rangle).
\]
Danciger: Assume \([., .]\) is a compact bilinear symmetric form in a complex Hilbert space \(H\). Let \(\sigma_0 \geq \sigma_1 \geq \sigma_2 \geq \cdots \geq 0\) be the singular values, repeated according to multiplicity. Then

\[
\min_{\text{codim} V = n} \max_{x \in V} \frac{\Re [x, x]}{\|x\|} = \sigma_{2n}
\]

\[
\sigma_n = 2 \min_{\text{codim} V = n} \max_{(x, y) \in V} \frac{\Re [x, y]}{\|(x, y)\|} = 1
\]

for all \(n \geq 0\). Here \(V\) denotes a \(\mathbb{C}\)-linear subspace of \(H \oplus H\).
Abstract Friedrichs Inequality

Danciger-Garcia-P.: Same conditions: $[.,.]$ compact, with spectrum $\sigma_k$ and eigenvalues $u_k$. Then

$$|[x,x]| \leq \sigma_2\|x\|^2$$

whenever $x$ is orthogonal to the vector $\sqrt{\sigma_1}u_0 + i\sqrt{\sigma_0}u_1$.

Furthermore, the constant $\sigma_2$ is the best possible for $x$ restricted to a subspace of $H$ of codimension one.
The ellipse

\[ \Omega_t - \text{the interior of the ellipse} \]

\[ \frac{x^2}{\cosh^2 t} + \frac{y^2}{\sinh^2 t} < 1, \]

where \( t > 0 \) is a parameter.

Quadrature identity

\[ \int_{\Omega_t} f(z) \, dA(z) = (\sinh 2t) \int_{-1}^{1} f(x) \sqrt{1 - x^2} \, dx \]
hence

$$\int_{\Omega_t} fg dA = (\sinh 2t) \int_{-1}^{1} f(x)g(x)\sqrt{1-x^2} \, dx.$$ 

Singular values $\sigma_n(t)$ and normalized (in $L^2_{a(\Omega_t)}$) singular vectors $e_n$:

$$\sigma_n(t) = \frac{(n+1) \sinh 2t}{\sinh[2(n+1)t]}$$

$$e_n(z) = \sqrt{\frac{2n+2}{\pi \sinh[2(n+1)t]}} U_n(z)$$

where $U_n$ denotes the $n$th Chebyshev polynomial of the second kind.
Since $U_0 = 1$, $U_1 = 2z$, and

$$
\sigma_2 = \frac{3 \sinh 2t}{\sinh 6t},
$$

we obtain

$$
\sqrt{\sigma_1} e_0(z) - i \sqrt{\sigma_0} e_1(z) = \frac{2}{\sqrt{\pi \sinh 4t}}(1 - 2iz),
$$

then the inequality

$$
\left| \int_{\Omega_t} f^2 \, dA \right| \leq \left( \frac{3 \sinh 2t}{\sinh 6t} \right) \int_{\Omega_t} |f|^2 \, dA
$$

holds whenever $f \perp (1 - 2iz)$. Furthermore, the preceding inequality is the best possible that can hold on a subspace of $L^2_a(\Omega_t)$ of codimension one.
Takagi’s work

Original approach to the Carathéodory-Fejér problem; leads to the form

\[ B(f, g) = \frac{(ufg)^{(n)}(0)}{n!}, \quad f, g \in H^2(\mathbb{T}). \]

where \( u(z) = c_0 + c_1 z + \cdots + c_n z^n \) is a prescribed Taylor polynomial at the origin.
Extremal problem

There exists an analytic function $F$ in the unit disk such that

$$F(z) = c_0 + c_1 z + \cdots + c_n z^n + O(z^{n+1})$$

and $\|F\|_\infty \leq M$ if and only if

$$\max_{\|f\|_2=1} \frac{1}{n!} \left| (uf^2)^{(n)}(0) \right| \leq M,$$

where $f$ is a polynomial of degree $\leq n$ and $\|f\|_2$ denotes the $l^2$-norm of its coefficients.
C-symmetric operators

Let $H$ be a Hilbert space with an anti-linear conjugation $C$, which is isometric: $\|Cx\| = \|x\|, C^2 = I$. An operator $T$ is called $C$-symmetric, if $T^*C = CT$, i.e. $T$ is symmetric w.r. to the form

$$\{x, y\} = \langle x, Cy \rangle.$$

Our examples were of the form

$$[x, y] = \langle Tx, Cy \rangle = \{Tx, y\}.$$
Polar decomposition

Garcia-P.: \( T \in L(H) \) is \( C \)-symmetric if and only if

\[
T = CJ|T|
\]

where \( J \) is another isometric anti-linear conjugation which commutes with \( |T| \).

Refines f. dim. decompositions of Takagi and Schur, and infinite dim. ones of Godic-Lucenko.
Compact C-sym. operators

\[ |T| u_k = \sigma_k u_k \]

admits \( J \)-invariant solutions, hence

\[ J|T| u_k = \sigma_k u_k, \]

and

\[ T u_k = C J|T| u_k = \sigma_k C u_k. \]

\[ \|T\| = \max\{\lambda \geq 0; \text{there exists } x \neq 0, \; Tx = \lambda Cx\}. \]
C-symmetric approximants

If $T = CJ|T|$ is compact $C$-symmetric, then choose $F_n \geq 0, F_n|T| = |T|F_n$, of rank $n + 1$, so that

$$\| |T| - F_n \| \leq \sigma_{n+1} = \sigma_{n+1}(|T|).$$

Then $T_n = CJF_n$ is $C$-symmetric, and

$$\| T - T_n \| \leq \sigma_{n+1}.$$
Unbounded operators

$H$ with isometric conjugation $C$

$T : \mathcal{D}(T) \rightarrow H$ closed graph, densely defined is $C$-symmetric, if

$$\mathcal{D}(T) \subset C\mathcal{D}(T^*)$$

and

$$CTC \subset T^*.$$ 

For instance Schrödinger operators with complex potentials, or certain PDE’s with non-symmetric boundary values are $C$-symmetric.
Example

Let $q(x)$ be a real valued, continuous, even function on $[-1, 1]$ and let $\alpha$ be a nonzero complex number satisfying $|\alpha| < 1$. For a small parameter $\epsilon > 0$, we define the operator

$$[T_\alpha f](x) = -if'(x) + \epsilon q(x)f(x),$$

with domain

$$\mathcal{D}(T_\alpha) = \{ f \in L^2[-1, 1] : f' \in L^2[-1, 1], f(1) = \alpha f(-1) \}.$$
If $C$ denotes the conjugation operator $[Cu](x) = \overline{u(-x)}$ on $L^2[-1,1]$, then it follows that that nonselfadjoint operator $T_\alpha$ satisfies $T_\alpha = CT_1/\alpha C$ and $T_\alpha^* = T_{1/\alpha}$ and hence $T_\alpha$ is a $C$-selfadjoint operator.

Takagi’s anti-linear equation

$$(T_\alpha - \lambda)u_n = \sigma_n Cu_n,$$

will give, for $n = 0$ the norm of the resolvent of $T_\alpha$. 
Application

\(-\nabla^2_D\) Laplace operator with zero (Dirichlet) boundary conditions over a finite domain (with smooth boundary) \(\Omega \subset \mathbb{R}^d\).

\(v(x) \geq 0\) be a scalar potential, which is \(\nabla^2_D\)-relatively bounded, with relative bound less than one.

\[H : \mathcal{D}(\nabla^2_D) \longrightarrow L^2(\Omega); \quad H = -\nabla^2_D + v(x),\]

the associated selfadjoint Hamiltonian with compact resolvent.
Assumption on $H$: its energy spectrum $\sigma$ consists of two parts, $\sigma \subset [0, E_-] \cup [E_+, \infty)$, which are separated by a gap $G \equiv E_+ - E_- > 0$.

Let $E \in (E_-, E_+)$ and $G_E = (H - E)^{-1}$ be the resolvent and take the average

$$ \bar{G}_E(x_1, x_2) \equiv \frac{1}{\omega_\epsilon^2} \int_{|x-x_1| \leq \epsilon} dx \int_{|y-x_2| \leq \epsilon} dy \ G_E(x, y), $$

where $\omega_\epsilon$ is the volume of a sphere of radius $\epsilon$ in $\mathbb{R}^d$. 
Green function estimate

Garcia-Prodan-P.: For q smaller than a critical value $q_c(E)$, there exists a constant $C_{q,E}$, independent of $\Omega$, such that:

$$|\tilde{G}_E(x_1, x_2)| \leq C_{q,E} e^{-q|x_1-x_2|}.$$ 

$C_{q,E}$ is given by

$$C_{q,E} = \frac{\omega_e^{-1} e^{2q e}}{\min |E_{\pm} - E - q^2|} \frac{1}{1 - q/F(q,E)}$$

with

$$F(q, E) = \sqrt{\frac{(E_+ - E - q^2)(E - E_- + q^2)}{4E_-}}.$$ 

The critical value $q_c(E)$ is the positive solution of the equation $q = F(q, E)$. 