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A HILBERT SPACE APPROACH TO BOUNDED ANALYTIC EXTENSION IN THE BALL

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Abstract. One proves, using methods of Hilbert spaces with a reproducing kernel, that any bounded analytic function on a complex curve in general position in the unit ball of $\mathbf{C}^n$ extends to a function in the Schur class of the ball.

1. Introduction.

Let Ω be a domain in $\mathbf{C}^n, n \geq 1$, and let V be a complex subvariety of Ω. While the extension of analytic function along V to Ω is a cohomological problem, and was solved by Cartan's Theorem B, the same question for bounded analytic functions is more delicate. Partial solutions to the latter extension problem were first provided by G. M. Henkin in the generic case [11], [12], [13], E. Amar [5] for non-generic subvarieties, and since then by many other authors, all relying on integral representation formulas or uniform estimates for the solutions of the Cauchy-Riemann system of equations. Along the same lines, T. Oshawa has recently proved remarkable extension theorems with L^2 bounds [15]. A fine account of these extension results can be found in the survey article [1].

On the other hand, in the case of a single complex variable, simple and powerful tools of Hilbert spaces with a reproducing kernel have succesfully been used in bounded analytic interpolation. These methods have also interacted with classical function theory and had far reaching consequences for the control theory of linear systems. Among the many references on the subject we cite [2], [3] and [10]. Only in

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recent time this framework has started to ramify towards several complex variables [2], [4], [6], [8]. One notable exception being the early paper [14].

The aim of the present note is to illustrate, in one of the simplest generic configurations in the unit ball of $\mathbf{C}^n$, how this classical Hilbert space approach can lead to and improve bounded analytic extension results. The possibly new method we propose below (in the context of several complex variables) consists of two steps of independent interest: the identification, as a set, of a Hilbert space with reproducing kernel with a classical space on the domain, such as the Hardy or Bergman space, and second the realization of the analytic multipliers on this space as the characteristic function of a tuple of operators linked to the geometry of the domain. Then the analytic extension is derived from this operator realization of the given data. The second step is already well understood and explained in the cited works. The first step remains however at an early stage. As mentioned before, these ideas are by now well established in the theory of functions of a single complex variable.

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2. **Main result.** First we recall some terminology and basic facts. Let B denote the unit ball in $\mathbf{C}^n, n \geq 1$, with complex coordinates $z = (z_1, \ldots, z_n)$ and Hermitian scalar product $\langle z, w\rangle = \sum_{j=1}^n z_1\overline{w}_1 + \ldots + z_n\overline{w}_n$. The Hardy space $H^2(B)$ will be regarded either as a Hilbert space of analytic functions in B with reproducing kernel $K(z,w) = \frac{1}{(1-\langle z,w\rangle)^n}$, or as the closure in $L^2(\partial B, d\sigma)$ of all complex polynomials, where $d\sigma$ is the area measure of the unit sphere. It is easy then to identify the algebra $H^\infty(B)$ of all bounded analytic functions in B with the class of all linear bounded operators on $H^2(B)$ which commute with the multiplication operators by the variables $z_j, 1 \leq j \leq n$.

It is well known that, in dimension greater than one, the classical Nevanlinna-Pick interpolation problem can constructively be solved by positivity conditions only in the smaller Schur class of functions. This is the set $S(B)$ of those analytic functions f on B which are bounded multipliers of the Hilbert space $H(k)$ with reproducing kernel:

$$k(z,w) = \frac{1}{1-\langle z,w\rangle}, \quad z, w \in B.$$

Or equivalently, such that the kernel:

$$\frac{M^2 - f(z)\overline{f(w)}}{1-\langle z,w\rangle}, \quad z, w \in B,$$

is non-negative definite. The best positive constant M above is the Schur norm $\|f\|_S$ of the function f. By restricting the above positive kernel to the diagonal we obtain the one sided estimate:

$$\|f\|_\infty \leq \|f\|_S.$$

One can prove that, for $n > 1$, the two norms are not equivalent, to the extent that $S(B)$ is a proper subspace of $H^\infty(B)$. For further details see [2], [4], [6], [8] and the references cited there to the fast growing literature on the Schur class.

The following result, proved for instance in [2], can be considered as the correct analogue, in the unit ball of $\mathbf{C}^n$, of the Nevanlinna-Pick interpolation theorem.

THEOREM 2.1. *Let A be a subset of the unit ball $B \subset \mathbf{C}^n$, and let $f : A \longrightarrow \mathbf{C}$ be a function. If the kernel:*

$$\frac{1 - f(z)\overline{f(w)}}{1 - \langle z, w \rangle}, \quad z, w \in A,$$

is non-negative definite, then there exists an analytic function $F \in S(B)$ with the properties:

$$\|F\|_S \leq 1, \quad F|_A = f.$$

It is interesting to note that, in the above statement, if the set A contains a complex submanifold V of B, then the positivity condition implies that the function f is analytic on V.

We would like to use the above general theorem in the case when A is a complex curve in B. More specifically, by an *analytic disk* in B attached to the unit sphere we mean an injective analytic map from the unit disk $\phi : \mathbf{D} \longrightarrow B$, extendable of class C^1 to the closure $\overline{\mathbf{D}}$, and such that:

$$\|\phi(u)\| = 1 \ \Leftrightarrow |u| = 1.$$

We will assume the transversality condition:

$$\langle \phi'(u), \phi(u) \rangle \neq 0, \quad |u| = 1. \tag{1}$$

Note that the gradient of ϕ may vanish at interior points of the disk, which means that the set A can have isolated singularities in the ball, but, due to the injectivity of f, no cross-intersections.

The map $\phi : \mathbf{D} \longrightarrow B$ induces by pull-back a positive definite kernel on the disk:

$$(\phi^* k)(u, v) = \frac{1}{1 - \langle \phi(u), \phi(v) \rangle}, \quad u, v \in \mathbf{D}.$$

Let $H(\phi^* k)$ be the associated reproducing kernel Hilbert space.

PROPOSITION 2.1. *With the above notation and assumptions $H(\phi^* k) = H^2(\mathbf{D})$.*

Consequently the bounded analytic mutipliers of the two spaces coincide with $H^\infty(\mathbf{D})$. In other terms, given a bounded analytic function $f \in H^\infty(A)$, there exists a positive constant M, with the property that the kernel:

$$\frac{M^2 - f(\phi(u))\overline{f(\phi(v))}}{1 - \langle \phi(u), \phi(v) \rangle}, \quad u, v \in \mathbf{D}, \tag{2}$$

is non-negative definite. In view of Nevanlinna-Pick's type Theorem 2.1, the above positivity condition implies that f has a bounded analytic extension to the Schur class of the ball. This proves the following theorem.

THEOREM 2.2. *Let A be an analytic disk in the unit ball of $\mathbf{C}^n$, transversally attached to the unit sphere. Then any bounded analytic function on A admits analytic extensions to the Schur class of the ball.*

3. **Proof of proposition.** It remains to prove Proposition 2.2. According to Banach's open mapping principle, we have to prove that there are positive constants C_1, C_2 with the property:

$$C_1 \sum_{j,k=1}^{d} \frac{\lambda_j \overline{\lambda}_k}{1 - u_j \overline{u}_k} \leq \sum_{j,k=1}^{d} \frac{\lambda_j \overline{\lambda}_k}{1 - \langle \phi(u_j), \phi(u_k) \rangle} \leq C_2 \sum_{j,k=1}^{d} \frac{\lambda_j \overline{\lambda}_k}{1 - u_j \overline{u}_k}, \tag{3}$$

for all finite sets of points $u_1, u_2, \dots, u_d \in \mathbf{D}$ and complex numbers $\lambda_1, \lambda_2, \dots, \lambda_d \in \mathbf{C}$.

Since the statement is invariant under linear fractionar transformations which map the ball into itself, we can assume that $\phi(0) = 0$. Thus, for every multiindex α, the scalar function ϕ^α vanishes of the order $|\alpha|$ or higher, at $z = 0$.

Let $d\mu(u) = \frac{du}{2\pi i \; u}$ be the normalized arc length masure on the unit circle T. The above estimates are equivalent to:

$$C_1 \|f\|^2_{2,T} \leq \int_T \int_T \frac{\overline{f(u)} f(v) d\mu(u) d\mu(v)}{1 - \langle \phi(u), \phi(v) \rangle} \leq C_2 \|f\|^2_{2,T}, \tag{4}$$

for any rational function f with poles outside the closed unit disk.

Let us introduce the linear operator:

$$(Rf)(u) = \int_T \frac{f(v) d\mu(v)}{1 - \langle \phi(u), \phi(v) \rangle}.$$

Remark that its representing kernel $\phi^* k$ is Hermitian, and can be written as:

$$\phi^* k(u,v) = \frac{1}{1 - \langle \phi(u), \phi(v) \rangle} = \frac{v-u}{\langle \phi(v) - \phi(u), \phi(v) \rangle} \cdot \frac{1}{v-u}, \quad |u| < 1, |v| = 1.$$

By the transversality assumption (1) we infer that:

$$\sup_{|u|<1, |v|=1} \left| \frac{v-u}{\langle \phi(v) - \phi(u), \phi(v) \rangle} \right| < \infty,$$

thus the operator R can be represented as:

$$(Rf)(u) = \int_T \frac{L(u,v) f(v) d\mu(v)}{1 - u\overline{v}},$$

where L is a continuous function on $\overline{\mathbf{D}} \times T$, analytic in the first variable. Thus R is a symmetric, densely defined operator on the Hardy space $H^2(\mathbf{D})$. The very definition of the kernel $\phi^* k$ shows also that $R \geq 0$ in the operator sense.

Next we have to check that R is bounded from above and below. To this aim, we write the representing kernel of R as:

$$\phi^* k(u,v) = \frac{L(u,v)}{1 - u\overline{v}} = \frac{L(u,v) - L(v,v)}{1 - u\overline{v}} + \frac{L(v,v)}{1 - u\overline{v}},$$

and remark that:

$$\frac{L(u,v) - L(v,v)}{1 - u\overline{v}}, \quad |u| < 1, |v| = 1,$$

extends to a continuous function on $\overline{\mathbf{D}} \times T$, while

$$L(v,v) = \frac{\overline{v}}{\langle \phi'(v), \phi(v) \rangle},$$

is a continuous, invertible function on the unit circle T.

Therefore the operator R is bounded and Fredholm, as a compact perturbation of a Toeplitz operator with continuous, invertible symbol, [7]. Since the spectrum of R is discrete in a neighbourhood of zero, it remains to verify that $\ker R = 0$.

Assume that the function $f \in H^2(\mathbf{D})$ satisfies $Rf = 0$. For any value $u \in \mathbf{D}$, we find a convergent power series expansion:

$$\sum_{\alpha \in \mathbf{N}^n} \gamma_\alpha \phi(u)^\alpha \langle f, \phi^\alpha \rangle_{2,T} = 0.$$

Since $\phi(0) = 0$, for every polynomial $p(u)$, only finitely many of the scalar products $\langle p, \phi^\alpha \rangle$ are non-zero. In particular, for any polynomial $p(u)$ we obtain:

$$\langle Rp, p \rangle = \int_T \int_T \frac{\overline{p(u)} p(v) d\mu(u) d\mu(v)}{1 - \langle \phi(u), \phi(v) \rangle} =$$

$$\sum_{\alpha \in \mathbf{N}^n} \gamma_\alpha |\langle p, \phi^\alpha \rangle|^2.$$

Since the operator R is bounded, we can pass to the limit in the Hardy space and obtain:

$$\langle Rh, h \rangle = \sum_{\alpha \in \mathbf{N}^n} \gamma_\alpha |\langle h, \phi^\alpha \rangle|^2, \quad h \in H^2(\mathbf{D}). \tag{5}$$

Returning now to the element $f \in \ker R$ we infer:

$$\langle f, \phi^\alpha \rangle = 0, \quad \alpha \in \mathbf{N}^n.$$

To reach the conclusion $f = 0$, it is sufficient to prove that the scalar functions $\phi(u)^\alpha$, $\alpha \in \mathbf{N}^n$, span a dense subset of $H^2(\mathbf{D})$.

Let $A(\mathbf{D})$ be the Banach algebra of all analytic functions in $\mathbf{D}$ which admit continuous extensions to $\overline{\mathbf{D}}$. Each entry ϕ_k of the map ϕ belongs to $A(\mathbf{D})$, and it can be approximated in this space by a sequence of functions which are analytic in a neighbourhood of $\overline{\mathbf{D}}$.

Thus, for our density question, we can assume that the map ϕ analytically extends to a neighbourhood of $\overline{\mathbf{D}}$, and remains injective there. Then, by Cartan's Theorem B for instance, the function $\phi^{-1} : A \longrightarrow \mathbf{D}$ analytically extends to a function $F : \overline{B} \longrightarrow \mathbf{C}$. In other terms, a series decomposition like:

$$u = F(\phi(u)) = \sum c_\alpha \phi^\alpha(u),$$

exists and is uniformly convergent on $\overline{\mathbf{D}}$. Consequently, all complex polynomials are in the closure in $A(\mathbf{D})$ of $\phi(u)^\alpha$, $\alpha \in \mathbf{N}^n$. In particular, this system of functions is dense in $H^2(\mathbf{D})$ and the proof is complete.

4. **Constructive aspects.** In this section we briefly discuss, under the assumptions of Theorem 2.3, how to construct the analytic extension and how to estimate its norm. Assume, as in Theorem 2.3, that the analytic function f is given on the analytic disk $A \subset B$. Then we know by Proposition 2.2 that there exists a positive constant M, such that the kernel (2) is non-negative definite.

The canonical construction of the analytic extension $F \in S(B)$ is obtained as follows (see [2]). Since the kernel (2) is non-negative definite, there exists an auxiliary Hilbert space K (with inner product denoted as $[x, y]$) and a map $\tau : A \longrightarrow K$, such that:

$$\frac{M^2 - f(\phi(u))\overline{f(\phi(v))}}{1 - \langle \phi(u), \phi(v) \rangle} = [\tau(\phi(u)), \tau(\phi(v))], \quad u, v \in \mathbf{D}.$$

We consider the Hilbert space $\mathbf{C} \oplus K^n$, and, for every point $z \in A$, the correspondence:

$$V \begin{pmatrix} M \\ z_1 \tau(z) \\ z_2 \tau(z) \\ \vdots \\ z_n \tau(z) \end{pmatrix} = \begin{pmatrix} f(z) \\ \tau(z) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

By its construction V is an isometry, hence it extends to the linear span of the vectors in its domain. Thus, V can be extended to a (linear) isometry W defined on the whole space $\mathbf{C} \oplus K^n$. Let us write it as:

$$W = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $B = (B_1, B_2, ..., B_n) : K^n \longrightarrow K$. Denote by $z\tau(z)$ the column vector of entries $z_j\tau$, $1 \leq j \leq n$, and by $\sigma(z)$ the colum vector of entries $(\tau(z), 0, \ldots, 0)$. Finally let P be the projection of K^n onto its first factor.

With these notation, the block matrix W satisfies, for every $z \in A$:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} M \\ z\tau(z) \end{pmatrix} = \begin{pmatrix} f(z) \\ \sigma(z) \end{pmatrix}.$$

An elementary elimination leads to the identity:

$$f(z) = M(A \cdot 1 + Bz(I - PDz)^{-1}PC \cdot 1), \quad z \in A.$$

Finally, the right hand side expression makes sense for $z \in B$ and it defines there an analytic function of Schur norm not exceeding M. For more details see [2].

To estimate the size of the constant M above, we can assume that $\|f\|_{\infty,A} = 1$. Let $g = f \circ \phi \in H^\infty(\mathbf{D})$, regarded as a bounded multiplier on the spaces $H^2 = H(\phi^* k)$, as in Proposition 2.2. Let $h \in H(\phi^* k)$ be an element of norm one, in short $\|h\|_k = 1$. Then,

$$\|gh\|_k \leq C_2\|gh\|_2 \leq C_2\|g\|_\infty\|h\|_2 \leq \frac{C_2}{C_1}\|g\|_\infty.$$

On the other hand, the operator $R : H^2(\mathbf{D}) \longrightarrow H^2(\mathbf{D})$ introduced in the proof of Proposition 2.2 has the norm $\|R\| = C_2$ and C_1 as the lowest eigenvalue.

In conclusion we have proved the following result.

Proposition 4.1. *Let $\phi : \mathbf{D} \longrightarrow B$ be an analytic disc, transversally attached to the unit sphere. Let $R : H^2(\mathbf{D}) \longrightarrow H^2(\mathbf{D})$ be the positive integral operator:*

$$(Rf)(u) = \frac{1}{2\pi i} \int_T \frac{f(v)\bar{v}dv}{1 - \langle \phi(u), \phi(v) \rangle}.$$

Denote by $\lambda(R)$ is lowest eigenvalue.

Then every bounded analytic function f on A extends to a function $F \in S(B)$ satisfying:

$$\|F\|_\infty \leq \|F\|_S \leq \frac{\|R\|}{\lambda(R)}\|f\|_{\infty,A}.$$

The case $\lambda(R) = \|R\|$ corresponds to a linear isometric embedding ϕ. It is known that only in this situation there exists an analytic extension of f which does not increase its uniform norm, see [9] or [17].

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