POSITIVE POLYNOMIALS ON PROJECTIVE LIMITS OF REAL ALGEBRAIC VARIETIES

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ABSTRACT. We reveal some key geomteric aspects related to non-convex optimization of sparse polynomials. The main result, a Positivstellensatz on the fibre product of real algebraic, affine varieties, is iterated to a comprehensive class of projective limits of such varieties. This framework includes as necessary ingredients recent works on the multivariate moment problem, disintegration and projective limits of probability measures and basic techniques of the theory of locally convex vector spaces. A variety of applications illustrate the versatility of this novel geometric approach to polynomial optimization.

1. INTRODUCTION

The ubiquous duality between ideals and algebraic varieties is replaced in semi-algebraic geometry by a duality between pre-orders, or quadratic modules in a ring (see the preliminaries for the exact definitions) and their positivity sets. This is already a non-trivial departure from classical algebraic geometry, well studied and understood only in the last decades with tools from real algebra, logic and functional analysis, see [26] for a recent, updated introduction. Significant applications to polynomial optimization have recently emerged from such

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abstract studies in real algebraic geometry, see for instance the survey [10].

Inspired by some recent, far reaching results in the optimization of polynomials with a sparse pattern in their coefficients, advocated by Kojima, Lasserre and their collaborators, we propose in this article a general framework for Striktpositivstellensätze on convex cones which are more general than the well studied preorders or quadratic modules. Our approach is based on the following geometric construction:

Let $f_i : X_i \longrightarrow Y,\ i = 1, 2$, be two morphisms of real algebraic, affine varieties, and let $X_1 \times_Y X_2$ be the (reduced) fibre product, with projection maps $u_i : X_1 \times_Y X_2 \longrightarrow X_i$. Given convex cones $C_i \subset \mathbb{R}[X_i]$ with positivity sets $K(C_i) \subset X_i$, we provide algebraic certificates for elements of $u_1^*\mathbb{R}[X_1]+u_2^*\mathbb{R}[X_2]$ to be positive on $u_1^{-1}K(C_1)\cap u_2^{-1}K(C_2)$.

By iterating this construction over a certain class of oriented graphs we incorporate in our geometric framework the main results circulating in optimization theory [15, 23, 20, 21, 33] and provide new applications, for instance to global optimization on unbounded sets.

The present paper unifies and extends in a natural way the recent results of the first author [17, 18] and some older observations and methods due to Schmüdgen [31] and the second author [27].

Our proofs use a separation of convex sets by linear functionals. And when dealing with functionals which are non-negative on convex cones of squares of polynomials, representing them by positive measures is a most desired objective. Thus, via this avenue, we are led to disintegration phenomena and existence of projective limits of probability measures. The exsitent results (some of them classical, such as Kolmogorov-Bochner-Prokhorov theorem on the existence of projective limits of probability measures) play a key, complementary role to our geometric study.

In this way, in particular, we can treat with the same techniques holomorphic functions of an infinity of variables, or positivity of polynomials on non-semi-algebraic sets.

The last part contains an abundance of examples, some of them dealing with classical fibre products, others with unbounded supporting sets of positivity, others with group actions or trigonometric polynomials. We have not touched in this article, but plan to do it in a future one, the natural and very possible extension to hermitian sums of squares and several complex variables.

The rather lengthy, inhomogeneous but necessary preliminaries make difficult a linear reading of this article. We propose the reader to start with the section containing the main results and fill the needed information with preliminaries in the way. However, among the latter, there are some results of independent interest, such as Theorem 3.1, wich can be regarded as a truncated version of a classical theorem of Haviland and as a generalization of a theorem of Tchakaloff, see for instance [29] for details.

2. Preliminaries

2.1. **Quadratic modules and Positivestellensätze.** Let A be a commutative ring with 1. For simplicity we assume that $\mathbb{Q} \subset A$. A *quadratic module* $Q \subset A$ is a subset of A such that $Q + Q \subset Q, 1 \in Q$ and $a^2 Q \subset A$ for all $a \in A$. We denote by $Q(M; A)$ the quadratic module generated in A by the set M. That is $Q(M; A)$ is the smallest subset of A which is closed under addition and multiplication by squares a^2, $a \in A$, containing M and the unit $1 \in A$. If M is finite, we say that the quadratic module is finitely generated. A quadratic module which is also closed under multiplication is called a *quadratic preordering.*

In the terminology used throughout this note, a real algebraic, affine variety $X \subset \mathbb{R}^d$ is the common zero set of a finite set of polynomials, and the algebra of regular functions on X is $A = \mathbb{R}[X] = \mathbb{R}[x_1, ..., x_d]/I(X)$, where $I(X)$ is the radical ideal associated to X. The non-negativity set of a subset $S \subset \mathbb{R}[X]$ is $K(S) = \{x \in X; f(x) \geq 0, \quad f \in S\}$. The duality between finitely generated quadratic modules and non-negativity sets plays a similar role in semi-algebraic geometry to the classical pairing between ideals and algebraic varieties, see [26] for details. A quadratic module $Q \subset \mathbb{R}[X]$ is *archimedean* if for every element $f \in \mathbb{R}[X]$ there exists a positive scalar α such that $1+\alpha f \in Q$. It is easy to see that the non-negativity set of an archimedean quadratic module is always compact.

The following Striktpositivestellensatz has attracted in the last decade a lot of attention from practitioners of polynomial optimization:

Theorem 2.1. *Let $Q \subset \mathbb{R}[x_1, ..., x_d]$ be an archimedean quadratic module and assume that a polynomial f is positive on $K(Q)$. Then $f \in Q$.*

The above fact was discovered by the second author [27], generalizing Schmüdgen's Striktpositivestellensatz [31] for the *finitely generated preordering* associated to a compact non-negativity set.

It is the aim of this present article to extend the above Striktpositivstellensatz to more general convex cones of polynomials defined on real algebraic varieties. Already several extensions in this direction are known ([17, 18, 19, 25, 32]), but, as explained in the introduction, the new impetus on decompositions of sparse polynomials serves as a motivation for our work.

2.2. **Fibre products of affine varieties.** Let X_1, X_2, Y be affine real varieties, endowed with the reduced structures, and let $f_i : X_i \longrightarrow Y,\ i = 1, 2,$ be morphisms. The (reduced) *fibre product* $X_1 \times_Y X_2$ is the affine sub-variety of $X_1 \times X_2$ consisting of those pairs $(x_1, x_2) \in X_1 \times X_2$ with $f_1(x_1) = f_2(x_2)$. The structural ring is the reduced of $\mathbb{R}[X_1] \otimes_{\mathbb{R}[Y]} \mathbb{R}[X_2]$. The morphisms $u_i : X_1 \times_Y X_2 \longrightarrow X_i,\ i = 1, 2,$ induced by the projections on the two factors close the diagram:

$$\begin{array}{ccc} X_1 \times_Y X_2 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & Y \end{array}$$

The fibre product can also be defined for schemes or more general bundles (known sometimes as the Whitney sum) and it always has a natural universality property, see for more details [9] Theorem 3.3 and [11] §1.4.

The main results of this article deal with regular functions defined on affine algebraic varieties obtained by repeated operations of fibre products. To put our constructions into a known setting we recall below some terminology related to projective limits (or sometimes called inverse limits) of varieties, or more general, topological spaces.

Let I be a non-empty set, endowed with a partial order relation $i \le j$. A *projective system* of algebraic varieties indexed over I consists of a family of varieties (affine in our case) $X_i,\ i \in I$, and morphisms $f_{ij} : X_j \longrightarrow X_i$ defined whenever $i \le j$, and satisfying the compatibility condition $f_{ik} = f_{ij} f_{jk}$ if $i \le j \le k$. The topological projective limit $X = \text{proj.lim}(X_i, f_{ij})$ is the universal object endowed with morphisms $f_i : X \longrightarrow X_i$ satisfying the compatibility conditions $f_i = f_{ij} f_j,\ \ i \le j$. See for instance [4] §6. A *directed projective system* carries the additional assumption on the index set that for every pair $i, j \in I$ there exists $k \in I$ satisfying $i \le k$ and $j \le k$. An oriented graph is canonically associated to the ordered set $(I, \le)$: its vertices are labelled by I and its arrows correspond to the order relation.

The fibre product is the projective limit of two arrows converging to the same target:

$$X_1 \xrightarrow{f_1} Y \xleftarrow{f_2} X_2.$$

When iterating this construction we obtain varieties of the form

$$(\ldots((X_1 \times_{Y_1} X_2) \times_{Y_2} X_3) \times_{Y_3} \ldots \times_{Y_n} X_{n+1}.$$

The associated ordered sets belong to the following category.

Definition 1. *Let $\mathcal{R}$ be the class of partially ordered sets $(I, \le)$ inductively constructed according to the following rules:*

(R_0) $\{\alpha_1, \alpha_2, \beta; \beta \le \alpha_2, \beta \le \alpha_2\} \in \mathcal{R}$;
(R_1) $(I, \le) \in \mathcal{R} \ \Rightarrow (I \cup \{\alpha, \beta\}; \exists i(\beta) \in I,\ i(\beta) \ge \beta,\ \ \alpha \ge \beta\}) \in \mathcal{R}$.

We do not exclude in the second axion $i(\beta) = \beta$, but we ask α to be an external element of I, so that what is called a rooted tree graph belongs to $\mathcal{R}$. On the other hand, the graph

$$* \longrightarrow * \longleftarrow * \longrightarrow * \longleftarrow \ldots \longrightarrow *$$

is another example of an ordered set belonging to $\mathcal{R}$.

2.3. **Projective limits of positive measures.** The proofs of the main results below use a duality argument. Since positive measures represent positive linear functionals, separating the polynomials, we will need certain constructions of projective limits of positive measures. Fortunately these constructions were performed and well understood a long time ago, due to applications to probability theory. We simply state the technical results we need. For more details and a general view of the probabilistic aspects we refer to the monographs by Bourbaki [4] and Bochner [3].

Lemma 1. *Let $Z = X_1 \times_Y X_2$ be a reduced fibre product of affine, real algebraic varieties, with structural maps $f_i : X_i \longrightarrow Y,\ u_i : Z \longrightarrow X_i,\ i = 1, 2$. Let μ_i be probability measures on $X_i,\ i = 1, 2$, respectively, satisfying $(f_1)_*\mu_1 = (f_2)_*\mu_2$..*

If the restricted maps $f_i : \mathrm{supp}\mu_i \longrightarrow Y$, are proper, then there exists a probability measure μ on Z satisfying $(u_i)_*\mu = \mu_i,\ i = 1, 2,$ and with $\mathrm{supp}\mu \subset [u_1^{-1}\mathrm{supp}\mu_1] \cap [u_2^{-1}\mathrm{supp}\mu_2]$.

Proof. A classical disintegration of measures theorem (see [4] Proposition IX.2.13) applies to both measures μ_i and gives:

$$\int h_i d\mu_i = \int_Y d\sigma(y) \int_{f_i^{-1}\{y\}} h_i(t) d\tau_y^i(t), \quad h_i \in \mathbb{R}[X_i],\ i = 1, 2,$$

where σ is a positive mesure on Y and τ_y^i are positive measures on the fibres $f_i^{-1}\{y\}$.

We define then

$$\int_Z h d\mu = \int_Y d\sigma(y) \int_{f_1^{-1}\{y\}\times f_2^{-1}\{y\}} h\ d\tau_y^1(t) \otimes d\tau_y^2(t).$$

By its very construction μ has push-forwards via the maps u_i to μ_i, and its support satisfies the inclusion in the statement. □

In case $X_1 = X_1' \times Y,\ X_2 = X_2' \times Y$ are affine spaces and f_1, f_2 are the projections onto Y maps, the above lemma provides the construction of a probability measure μ on $X_1' \times Y \times X_2'$, given its marginals on X_1 and X_2, see for instance [6].

There is a variety of exsitence theorems for infinite projective limits of probability measures. They require in general a directed inverse system, and some additional compactness assumptions on the structural maps (the so-called Prokhorov's condition). We state, for our aims, a couple of general results in this direction, see again for details [4].

Theorem 2.2. *Let (X_i, f_{ij}) be a directed projective system of topological spaces, and let μ_i be positive, finite measures on X_i, compatible with the structural maps: $(f_{ij})_*\mu_j = \mu_i$. Assume that, either*

a). the index set $(I, \leq)$ has a countable cofinal subset, or

b). $\mathrm{supp}\mu_i$ *is compact for all i.*

Then there exists a finite positive measure μ on $\mathrm{proj.lim}(X_i, f_{ij})$ *satisfying $(f_i)_*\mu = \mu_i,\ i \in I$.*

In the case of a countable product of probability spaces we recover in the above statement a classical theorem of Kolmogorov-Bochner asserting the existence of a probability measure on the infinite product space, with prescribed marginals.

2.4. **Sparse polynomials and chordal graphs.** Let $\mathbb{R}[x_1, ..., x_d]$ be the polynomial ring in d variables, and let $\{I_1, I_2, ..., I_k\}$ be a finite covering of the set of indices $I = \{1, 2, ..., d\}$. The polynomial ring in the variables I_j will be denoted in short $\mathbb{R}[x(I_j)]$ and the corresponding affine space by $\mathbb{R}^{I_j}$. The main question considered in polynomial optimization was to find certificates of positivity for a (sparse) element

$$f \in \mathbb{R}[x(I_1)] + ... + \mathbb{R}[x(I_k)]$$

subject to the conditions

$$(\forall \delta \in \Delta,\ g_\delta(x) \geq 0) \ \Rightarrow\ f(x) > 0,$$

where Δ is a finite set, each $g_\delta \in \mathbb{R}[I_{\phi(\delta)}]$ for a function $\phi : \Delta \longrightarrow \{1, ..., k\}$. Kojima and collaborators rightly realized that the above optimization problem can be put and relaxed in dual form, where it naturally leads to consider positive extensions of partially given multivariate moment matrices, cf [7, 23, 15, 16, 33], and for a slightly different, recent parallel study see also [20]. In its turn, the matrix completion problem with constraints was throughouly studied a few decades ago, see [12].

This positive matrix completion approach to optimization of sparse polynomials imposes a necessary and sufficient condition on the incidence graph attached to the sets of variables $\{I_1, I_2, ..., I_k\}$ known as the *chordal property*, see for details [23] and for general theory [12, 2]. The algorithmic aspects and computational complexity of grouping the variables into subsets satisfying the chordal property is analyzed in [23, 33] and we do not touch this territory. We only need the observation that, a covering $\{I_1, ..., I_k\}$ satisfying the chordal property can be rearranged so that, for all $j, 2 \leq j, \leq k$, there exists $k(j) < j$ such that

$$I_j \cap (I_1 \cup I_2 \cup ... \cup I_{j-1}) \subset I_{k(j)}. \tag{1}$$

This condition is known as the *running intersection property.* The relevance of this property for our work is summarized in the following obvious observation.

Proposition 1. *Let $\{I_1, ..., I_k\}$ be a covering of the set of indices $I = \{1, 2, ..., d\}$ satisfying the running intersection property (1). Then the partially ordered set underlying the projective system of affine spaces and projection maps*

$$\mathbb{R}^{I_{k(j)}} \longrightarrow \mathbb{R}^{I_{k(j)} \cap I_j} \longleftarrow \mathbb{R}^{I_j}, \quad 2 \leq j \leq k,$$

belongs to the class $\mathcal{R}$.

3. A general framework for algebraic certificates of positivity

In this section we recall some key concepts from the recent works of the first author [17, 18]. Specifically, let $X \subset \mathbb{R}^n$ be a real algebraic, affine variety, endowed with its reduced algebra $\mathbb{R}[X] = \mathbb{R}[x_1, ..., x_n]/I_X$ of regular functions. Let $\mathbb{R}_d[X]$ be the filtration induced by the degree: $p \in \mathbb{R}[X]$ if there exists $P \in \mathbb{R}[A^n]$, $\deg P \leq d$ and $p - P \in I_X$. For a subset $S \subset \mathbb{R}[X]$ we denote

$$S_d = S \cap \mathbb{R}_d[X].$$

When needed to make the distinction, we set $\mathbb{A}^n = \mathbb{R}^n$ and we refer to it as the real affine space of dimension n. Recall that for a subset $C \subset \mathbb{R}[X]$ we define its *positivity set* as

$$K(C) = \{x \in X; f(x) \geq 0, \quad f \in C\}.$$

Definition 2. *Let $C \subset \mathbb{R}[X]$ be a convex cone, and let V be the linear span of C in $\mathbb{R}[X]$. We say that C satisfies:*

(MP) (the moment property) if every linear functional $L \in V'$ which is non-negative on C is represented by a positive Borel measure supported on X;

(SMP) (the strong moment property) if every linear functional $L \in V'$ which is non-negative on C is represented by a positive Borel measure supported on $K(C)$;

(†) if $f \in V,\ f|_{K(C)} \geq 0$, then for every $\epsilon > 0, f + \epsilon \in C$;

(‡) if $f \in V,\ f|_{K(C)} \geq 0$, then there exists $q \in C$, such that, for every $\epsilon > 0, f + \epsilon q \in C$.

In the presence of a convex cone as above, we can analogously define a graded set of its closures:

$$C^{\text{lin}} = \{f \in V; L \in V', L|_C \geq 0 \Rightarrow L(f) \geq 0\},$$

$$C^{\dagger} = \{f \in V; f|_{K(C} \geq 0, \epsilon > 0 \ \Rightarrow \ f + \epsilon \in C\},$$

and

$$C^{\ddagger} = \{f \in V; f|_{K(C} \geq 0, \exists q \in C, \epsilon > 0 \ \Rightarrow \ f + \epsilon q \in C\}.$$

By its very definition, C^{lin} is the closure of C in the finest locally convex topology carried by V, that is that induced by the Euclidean norms of any finite dimensional subspace of V. The equality $C = C^{(\dagger)}$ holds for all archimedean quadratic modules C. The equality $C = C^{(\ddagger)}$ is more subtle and it will be analyzed below.

We do not discuss below at length the implications among the above conditions, or containments of the respective closures of C, simply referring to [17, 18] for full details in this direction. Instead, we focus on a couple of remarks directly related to the decomposition of sparse polynomials phenomena.

Lemma 2. *Let X be an affine, real algebraic variety, and let $C \subset \mathbb{R}[X]$ be a convex cone with V its linear span. Assume that for every positive integer d,*

$$\text{int} C_d \neq \emptyset. \tag{2}$$

Then

$$C^{(\ddagger)} = \cup_{d=1}^{\infty} \overline{C_d},$$

where both the interior and closure is taken in the Euclidean topology of V_d.

Proof. The proof is very similar to that of Proposition 1.3 in [17]. Namely, if $f \in C^{(\ddagger)}$, then $f = \lim_{\epsilon \to 0}(f + \epsilon q) \in \cup_{d=1}^{\infty} \overline{C_d}$. The other way around, if $f \in \overline{C_d}$, choose $q \in \text{int} C_d$, so that the straight line segment joining f to q lies in C_d. □

All quadratic modules or preorders with the archimedean property with respect to the algebra $\mathbb{R}[\mathbb{A}^n]$ satisfy property (‡), but there are interesting examples of non-archimedean cones with this property, as we shall see on some examples contained in the last section.

Let $X \subset \mathbb{A}^n$ be an affine, real algebraic variety, and let $K \subset X$ be a closed subset of X. We call after Reznick K *full*, if it is a uniqueness set in the algebra $\mathbb{R}[X]$, that is $p \in \mathbb{R}[X], p|_K = 0 \Rightarrow p = 0$. We denote by $\mathcal{P}_d^+(K) = \{p \in \mathbb{R}_d[X];\ p|_K \geq 0\}$. For d even, these are convex cones with non-empty interior in $\mathbb{R}_d[X]$ because, for M large,

$$2p(x) = (M\|x\|^d + p(x)) - (M\|x\|^d - p(x)),$$

whence $\mathcal{P}_d^+(K) - \mathcal{P}_d^+(K) = \mathbb{R}_d[X]$.

The next theorem represents a version of Haviland Theorem for the truncated moment problem, and in the same time it is a variation to Tchakaloff Theorem.

Theorem 3.1. *Let X be a real algebraic variety, and let $K \subset X$ be a closed, full subset. Fix an integer d and consider a linear functional $L \in \mathbb{R}_d[X]'$ satisfying $L|_{\mathcal{P}_d^+(K)} \geq 0,\ (f|_K \geq 0,\ L(f) = 0) \Rightarrow f = 0$. Then there exists a finite atomic measure supported by K representing L.*

Proof. We follow the ideas presented in [1], in the new proof of Tchakaloff Theorem.

Assume that $d = 1$ and consider the polar cone

$$\mathcal{P}_1^+(K)^\circ = \{\ell \in \mathbb{R}_1[X]'; \quad \ell|_{\mathcal{P}_1^+(K)} \geq 0\}.$$

By its definition $\mathcal{P}_1^+(K)$ is a closed convex cone in $\mathbb{R}_1[X]$, so it coincides with its bi-polar (see for instance [30]). Thus the assumption $(f|_K \geq 0,\ L(f) = 0) \Rightarrow f = 0$ means that L is not in the boundary of $\mathcal{P}_1^+(K)^\circ$, that is $L \in \mathrm{int}\mathcal{P}_1^+(K)^\circ$. By Minkowski's separation Theorem,

$$\mathcal{P}_1^+(K)^\circ = \overline{\mathrm{co}\{\delta_x;\ x \in K\}},$$

where δ_a stands for the Dirac measure at a. Consequently

$$L \in \mathrm{co}\{\delta_x;\ x \in K\},$$

which is the conclusion in the statement.

For $d > 1$ we list polynomial representatives $\{1, \phi_1, ..., \phi_N\}$ of the space $\mathbb{R}_d[X]$ and consider the Veronese type (proper) imbedding

$$(1, \phi_1, ..., \phi_N) : X \longrightarrow \mathbb{A}^{N+1}.$$

Then we apply the case $d = 1$ to the image of this map and the lift of the functional L there. □

For a slightly different proof and explanation of the existence of such cubature formulas see [29].

4. Main results

Let V be a real vector space and let C be a convex cone in V. We say that an element ξ belongs to the *algebraic interior*, in short $\xi \in \mathrm{alg.int}C$, if for every $f \in V$ there exists a positive constant λ such that $\xi + \lambda f \in C$. The following separation lemma (originally proved independently by Eidelheit, Kakutani and Krein) will be needed at a key technical point in our proofs.

Lemma 3. *Let $C \subset V$ be a convex cone in a real vector space V. Assume that $\xi \in \mathrm{alg.int}C$ and that $g \notin C$. Then there exists a linear functional $L \in V'$, such that*

$$L(g) \leq 0 \leq L(c), \quad c \in C; \quad L(\xi) = 1.$$

For a proof (a variation of the Hahn-Banach argument) see [14].

At this stage we are ready to assemble all ingredients into our main lemma.

Lemma 4. *(Basic Lemma) Let $X_1 \times_Y X_2$ be the reduced fibre product of affine, real algebraic varieties, with structural maps $f_i : X_i \longrightarrow Y$, $u_i : Z \longrightarrow X_i$, $i = 1, 2$.*

Let $C_i \subset \mathbb{R}[X_i]$ be convex cones with the (SMP) and $1 \in \text{alg.int}C_i$, with respect to the linear subspace V_i generated by C_i, $i = 1, 2$, respectively, and such that the maps $f_i : K(C_i) \longrightarrow Y$ are proper.

Assume that $p \in u_1^* V_1 + u_2^* V_2$ is positive on $u_1^{-1} K(C_1) \cap u_2^{-1} K(C_2)$. Then $p \in u_1^* C_1 + u_2^* C_2$.

Proof. Assume by contradiction that $p \notin u_1^* C_1 + u_2^* C_2$. Since $1 \in \text{alg.int}[u_1^* V_1 + u_2^* V_2]$, the separation lemma applies, and gives a linear functional $L \in [u_1^* V_1 + u_2^* V_2]'$, non-negative on $u_1^* C_1 + u_2^* C_2$ and satisfying $L(p) \leq 0 < L(1) = 1$. Let $L_i(f) = L(u_i^* f)$, $f \in V_i$, $i = 1, 2$. By assumption, L_i is represented by a probability measure μ_i, supported by the positivity set $K(C_i)$. Moreover

$$(f_1)_* \mu_1 = (f_2)_* \mu_2,$$

by definition:

$$\int g \ d(f_1)_* \mu_1 = L(u_1^* f_1^* g) = L(u_2^* f_2^* g) = \int g \ d(f_2)_* \mu_2.$$

Since the restricted maps $f_i : K(C_i) \longrightarrow Y$ are proper, Lemma 1 yields a positive measure μ supported by the set $S = u_1^{-1} K(C_1) \cap u_2^{-1} K(C_2)$, which represents the functional L. Consequently,

$$L(p) = \int_S p d\mu \leq 0,$$

a contradiction. □

The conditions in the statement are met if $C_i \subset \mathbb{R}[X_i]$ are quadratic modules with the archimedean property. Indeed, in this case $1 \in \text{alg.int}C_i$ and $K(C_i)$ are compact sets.

A repeated use of the basic lemma leads to our first main result. Without aiming at full generality, we state it in the case of archimedean quadratic modules. Variations on the same theme are straighforward.

Theorem 4.1. *Let (X_i, f_{ij}) be a finite projective system of real algebraic varieties, with the ordered index set belonging to the class $\mathcal{R}$. Assume that $Q_i \subset \mathbb{R}[X_i]$ are archimedean quadratic modules satisfying $(f_{ij})^* Q_i \subset Q_j$.*

An element $p \in \sum_i f_i^* \mathbb{R}[X_i]$ which is positive on $\cap_i f_i^{-1} K(Q_i)$ belongs to $\sum_i f_i^* Q_i$.

The coherence condition $(f_{ij})^* Q_i \subset Q_j$ implies $f_{ij} K(Q_j) \subset K(Q_i)$, that is we can actually work with the projective system $(K(Q_i), f_{ij})$ of compact spaces.

When dealing with unbounded positivity sets, the assumption $1 \in \text{alg.int} C_i$ is way too strong. Instead, we propose the following variation of the Basic Lemma.

Lemma 5. *Let $X_1 \times_Y X_2$ be the fibre product of affine, real algebraic varieties. Let $C_i \subset \mathbb{R}[X_i]$ be convex cones with*

$$\text{int}[C_i]_d \neq \emptyset \text{ in } [V_i]_d, \quad d \geq 1,$$

and satisfying the (‡) property, with respect to the vector subspaces $V_i \subset \mathbb{R}[X_i]$, $i = 1, 2$. Assume that the positivity sets $K(C_i)$ are full in X_i, and that the maps $f_i : K(C_i) \longrightarrow Y$ are proper. Let $f \in u_1^* V_1 + u_2^* V_2$ be non-negative on $u_1^{-1} K(C_1) \cap u_2^{-1} K(C_2)$. Then there exists $q \in u_1^* C_1 + u_2^* C_2$, such that, for all $\epsilon > 0$, $f + \epsilon q \in u_1^* C_1 + u_2^* C_2$.

Proof. By a simple variation in the proof of Lemma 2 we choose a large enough positive integer d, and assume by contradiction that

$$f \notin \overline{[(u_1^* C_1)_d + (u_2^* C_2)_d]}.$$

Note that the maps u_i are induced by projections, so $(u_i)^* f$ does not alter the degree of f, and in particular, if $\text{int}[C_i]_d \neq \emptyset$ with respect to $[V_i]_d$, $d \geq 1$, then $\text{int}[(u_1^* C_1)_d + (u_2^* C_2)_d] \neq \emptyset$, as a subspace of $[u_1^* V_1 + u_2^* V_2]_d$.

By Minkowski's Separation Theorem, there exists a linear functional $M \in [u_1^* V_1 + u_2^* V_2]'$ satisfying

$$M(f) < 0 \leq M|_{(u_1^* C_1)_d + (u_2^* C_2)_d}.$$

Since the positivity sets $K(C_i)$ are full in X_i, respectively, we can choose positive measures σ_i (of very small total mass), supported by $u_i^{-1}K(C_i)$, such that the functional

$$L(g) = M(g) + \int g[(u_1)_* d\sigma_1 + (u_2)_* d\sigma_2]$$

still satisfies

$$L(f) < 0 \leq L|_{(u_1^* C_1)_d + (u_2^* C_2)_d}$$

and in addition,

$$g|_{u_1^{-1}K(C_1) \cap u_2^{-1}K(C_2)} \geq 0, \quad L(g) = 0 \quad \Rightarrow g = 0.$$

According to Theorem 3.1 there are finite atomic, positive measures μ_i, supported by $K(C_i)$, respectively, so that

$$L(g_i \circ u_i) = \int_{K(C_i)} g_i d\mu_i, \quad g_i \in V_i, \ i = 1, 2.$$

Then the fibre product of measures lemma applies (Lemma 1), and we conclude that there exists a positive measure μ, supported by $u_1^{-1}K(C_1) \cap u_2^{-1}K(C_2)$, and such that

$$0 > L(f) = \int f d\mu.$$

A contradiction. □

Although not the most general statement, the theorem below gives the flavor of the kind of applications the above lemma opens.

Theorem 4.2. *Let $Z = X_1 \times_Y X_2$ be the fibre product of some real algebraic affine varieties, and let $Q_i \subset \mathbb{R}[X_i]$ be quadratic modules such that the induced maps $f_i : K(Q_i) \longrightarrow Y$ are proper.*

a). If both $Q_{1,2}$ have the (SMP), then $u_1^* Q_1 + u_2^* Q_2$ has the (SMP) with respect to the sparse vector space $u_1^*\mathbb{R}[X_1] + u_2^*\mathbb{R}[X_2]$;

b). If $K(Q_i)$ are full subsets of X_i, $i = 1, 2$, and both $Q_{1,2}$ have the (‡) property, then so does $u_1^* Q_1 + u_2^* Q_2$ with respect to the sparse vector space $u_1^*\mathbb{R}[X_1] + u_2^*\mathbb{R}[X_2]$.

Proof. The proof of a) is a direct application of Lemma 1, in the spirit of the proof of the Basic Lemma above.

To prove assertion b) we use Lemma 5, and the fact that quadratic modules satisfy the non-empty relative interior condition in the statement, see [17] Remark 1.1. □

For part a) the quadratic modules can be replaced by convex cones. However, for assertion b), if $Q_i = C_i$ are simply convex cones, then the assumption " non-empty interior" (2) is required.

Assertion a), plus the observation that the assumption on proper mappings is compatible with the iteration of fibre products leads to the following observation.

Corollary 4.3. *Let (X_i, f_{ij}) be a finite projective system of real algebraic varieties, with the ordered index set belonging to the class $\mathcal{R}$. Assume that $C_i \subset \mathbb{R}[X_i]$ are convex cones with the (SMP), satisfying $(f_{ij})^* C_i \subset C_j$ and such that the structural maps $f_{ij} : K(C_j) \longrightarrow K(C_i)$ are proper.*

Then the convex cone of sparse elements $\sum_i f_i^* C_i$ has the (SMP) with respect to the space $\sum_i f_i^*(\text{lin.span} C_i)$.

The analogue for assertion b) seems to be more delicate, due to the fullness assumption.

5. Infinite projective limits

We exemplify below how the existence of projective limits of probability measures gives natural decompositions of sparse polynomials (or limits of them) in the case of infinitely many variables or countably many constraints.

To fix ideas, let $X = \text{proj.lim}(X_i, f_{ij})$ be a directed projective limit of affine real varieties. Let $C_i \subset \mathbb{R}[X_i]$ be convex cones with compact positivity sets $K(C_i)$ and with the strong moment property (SMP). Assume that

$$f_{ij}^* C_i \subseteq C_j, \quad i \leq j.$$

That is

$$f_{ij} K(C_j) \subset K(C_i), \quad i \leq j,$$

whence $(K(C_i), f_{ij})$ forms a projective system of compact topological spaces. We consider on the space $Z = \text{proj.lim}(K(C_i), f_{ij})$ the restriction of the vector space of polynomial functions $\mathcal{P} = \sum_i {f_i}^* \mathbb{R}[X_i]$ and the closure of it $\overline{\mathcal{P}}$ in the product topology of uniform convergence on the factors $K(C_i)$.

Proposition 2. *In the above conditions, an element of $\overline{\mathcal{P}}$ which is non-negative on Z belongs to $[\sum_i f_i^* C_i]^-$.*

For the proof we again assume by contradiction that the element $f \in \overline{\mathcal{P}}$ does not belong to the closed cone in the statement. By Minkovski's separation theorem, there exists a linear functional L on $\overline{\mathcal{P}}$ which satisfies

$$L(f) < 0 \leq L(g), \quad g \in \overline{\mathcal{P}}.$$

In particular the restricted functionals $L : f_i^* \mathbb{R}[X_i] \longrightarrow \mathbb{R}$ are non-negative on every cone C_i, $i \in I$. Thus, by the assumption of the strong moment property, there are positive measures μ_i, supported on $K(C_i)$. By their very construction these measures are compatible with the projective system: $(f_{ij})_* \mu_j = \mu_i$. According to Theorem 2.2 there exists a positive measure μ on $\overline{\mathcal{P}}$, hence supported by Z, and representing L. In particular

$$0 > L(f) = \int_Z f d\mu \geq 0,$$

a contradiction.

6. Examples and applications

6.1. **The fibre product of reduced schemes may not be reduced.** Here is a simple example: consider $X_1 = X_1 = \mathbb{R}$, $Y = \mathbb{R}^2$ and the morphisms $f_1(x_1) = (x_1^2, -x_1^2)$, $f_2(x_2) = (x_2^2, x_2^2)$. Then the algebraic fibre product is the single point $X_1 \times_Y X_2 = \{(0,0)\} \subset \mathbb{R}^2$ with the nilpotent structure $\mathbb{R}[x_1, x_2]/(x_1^2, x_2^2)$. The reduced fibre product (which we use throughout this article), is the single point $\{(0,0)\}$ with the reduced ring of functions defined on it $\mathbb{R} = \mathbb{R}[x_1, x_2]/(x_1, x_2)$.

6.2. **Kojima-Lasserre Theorem.** Let $\{I_1, ..., I_k\}$ be a covering of the set of indices $I = \{1, 2, ..., d\}$ satisfying the running intersection property (1). Consider archimedean quadratic modules $Q_j \subset \mathbb{R}[x(I_j)], 1 \leq j \leq k$. In particular the positivity sets $K(Q_j)$ are compact (semi-algebraic) in $\mathbb{R}^{I_j}$. Let $\mathbf{K}(\mathbf{Q_j}) = K(Q_j) \times \mathbb{R}^{I \setminus I_j}$ be the associated cylinders in $\mathbb{R}^I$.

Theorem 6.1. *(Kojima-Lasserre) If $f \in \mathbb{R}[x(I_1)] + ... + \mathbb{R}[x(I_k)]$ is strictly positive on $\cap_{j=1}^k \mathbf{K}(\mathbf{Q_j})$, then $f \in Q_1 + Q_2 + ... + Q_k$.*

For the proof we simply invoke our main result Theorem 4.1 and Proposition 1. The original proofs have been published in [7, 23, 20, 21]. A completely different approach, self contained and very ingenious appears in [8].

6.3. **Sparsity with algebraic mixing.** We consider a few simple examples derived from Theorem 4.1.

Let $X_1 = X_2 = Y = \mathbb{R}^2$ with variables $(x_i, y_i) \in X_i,\ i = 1, 2$. Let $R_1(x_1, y_1), R_2(x_2, y_2)$ be non-constant polynomials, and let

$$f_1(x_1, y_1) = (R_1(x_1, y_1), y_1), \quad f_2(x_2, y_2) = (R_2(x_2, y_2), y_2).$$

The fibre product $Z = X_1 \times_Y X_2$ is contained in $\mathbb{R}^3$, with coordinates (x_1, y, x_2), and it is given by the equation

$$R_1(x_1, y) = R_2(x_2, y).$$

Let $Q_i \mathbb{R}[X_i]$ be archimedean quadratic modules. The lifting in Z of their positivity sets is

$$Z_+ = \{(x_1, y, x_2);\ (x_1, y) \in K(Q_1),\ (x_2, y) \in K(Q_2)\}.$$

Our main result then applies, and it can be stated as:

If a polynomial $p \in \mathbb{R}[x_1, y] + \mathbb{R}[y, x_2]$ is positive on Z_+, then there are elements $q_i \in Q_i$, so that

$$(R_1(x_1, y) = R_2(x_2, y)) \ \Rightarrow\ (p(x_1, y, x_2) = q_1(x_1, y) + q_2(x_2, y)).$$

To be even more specific, choose $R_1(x_1, y) = x_1 - h(y)$ so that we can solve explicitly the defining equation of the fibre product and find $Z = \mathbb{R}^2$, with coordinates (y, x_2). Then the statement becomes:

If the polynomial $p \in \mathbb{R}[x_1, y] + \mathbb{R}[y, x_2]$ *satisfies*

$$[(R_2(x_2, y) + h(y), y) \in P(Q_1))\&((x_2, y) \in P(Q_2))] \quad \Rightarrow$$

$$p(R_2(x_2, y) + h(y), y, x_2) > 0,$$

then

$$p(R_2(x_2, y) + h(y), y, x_2) = q_1(R_2(x_2, y) + h(y), y) + q_2(x_2, y),$$

with $q_i \in Q_i$.

6.4. **Vector bundles.** Assume that Y is an affine, real algebraic variety and that E_1, E_2 are vector bundles over Y, so that the total spaces X_1, X_2 are also affine varieties, choosing the maps $f_{1,2}$ to be the canonical projections. Then $X_1 \times_Y X_2$ is the total space of the Whitney sum $E_1 \oplus E_2$.

This is the case for instance of the tautological bundle $E_{1,2} = O(-1)$ over the sphere $Y = S^{d-1} \subset \mathbb{R}^d$ with total space $X_{1,2} = \{(x, v),\ x \in S^{d-1}, v \in \mathbb{R}^d, \exists\ \lambda \in R,\ v = \lambda x\}$.

A typical application of our main result reads as follows. *Let* $p(x, z_1, z_2) \in \mathbb{R}[x, z_1] + \mathbb{R}[x, z_2]$ *be a sparse polynomial, where* x, z_1, z_2 *are* d*-tuples of variables. If*

$$p(x, \lambda_1 x, \lambda_2 x) > 0 \quad \text{whenever} \|x\| = 1 \text{ and } |\lambda_i| \leq 1,\ i = 1, 2,$$

then there are sums of squares of polynomials σ_* *satsifying:*

$$(\|x\| = 1,\ z_1 = \lambda_1 x,\ z_2 = \lambda_2 x) \ \Rightarrow$$

$$p(x, z_1, z_2) = \sigma_1(x, z_1) + (1 - \|z_1\|^2)\sigma_2(x, z_1) + \sigma_3(x, z_2) + (1 - \|z_2\|^2)\sigma_4(x, z_2).$$

There are quite a few other "canonical" examples of vector bundles whose total space is a real algebraic variety. Take for instance a smooth, real algebraic variety hypersurface $X \subset \mathbb{R}^d$ given by a single polynomial equation

$$X = \{x \in \mathbb{R}^d;\ \rho(x) = 0\},$$

where we assume $\nabla\rho \neq 0$ along X. Then the tangent bundle

$$TX = \{(x, v) \in X \times \mathbb{R}^d;\ \nabla_x\rho \cdot v = 0\}$$

is real algebraic. We endow TX with the Riemannian metric given by the imbedding $X \subset \mathbb{R}^d$. Let

$$S(TX) = \{(x, v) \in X \times S^{d-1};\ \nabla_x\rho \cdot v = 0\}$$

be the associated bundle in spheres.

The fibre product of two copies of $TX \longrightarrow X$ is the total space of $(TX \oplus TX) \longrightarrow X$, that is the variety

$$TX \times_X TX = \{(x, v, w) \in X \times \mathbb{R}^d \times \mathbb{R}^d; \nabla_x\rho \cdot v = 0, \nabla_x\rho \cdot w = 0.\}$$

Without offering more details, we consider a situation relevant to the theory of linear elliptic PDE's. Namely, let

$$p(x, v, w) = p_1(x, v) + p_2(x, w)$$

be a polynomial, with p_1 homogeneous of degree $2m$ in v, and similarly $p_2(x, w)$ homogeneous of degree $2n$ in w. Assume that

$$(x \in X, \|v\| = \|w\| = 1)\ \Rightarrow\ p(x, v, w) > 0.$$

Then Theorem 4.1 applies, giving the decomposition

$$p(x, v, w) = \sigma_1(x, v) + \sigma_2(x, w) + (1 - \|v\|^2)\sigma_3(x, v) + (1 - \|w\|^2)\sigma_4(x, w),$$

where σ_* are sums of squares of polynomials in the respective variables. One step further, since p is homogeneous in both v, w, by restricting the above decomposition to the spheres, and compensating the even degrees by powers of $\|v\|^2, \|w\|^2$ we obtain a more precise information:

If the bi-homogeneous polynomial $p(x, v, w) = p_1(x, v) + p_2(x, w)$ is positive on $S(TX) \times_X S(TX)$, then there exists a positive integer k, such that

$$(x \in X)\ \Rightarrow\ (\|v\|^{2k} p_1(x, v) + \|w\|^{2k} p_2(x, w) = \sigma_1(x, v) + \sigma_2(x, w)),$$

where $\sigma_{1,2}$ are homogeneous in the second variable, and sums of squares.

6.5. **Homogeneous bundles.** Among the many classical homogeneous bundles (see for instance [11]) we first draw attention to the frame bundle on an embedded hypersurface.

Let $X = \{x \in \mathbb{R}^d;\ \rho(x) = 0\}, (x \in X) \ \Rightarrow \nabla_x \rho \neq 0$ as in the previous example and let $FX = \{(x, v_1, ..., v_{d-1}); \nabla_x v_i = 0,\ v_1 \wedge v_2 \wedge ... \wedge v_{d-1} \neq 0\}$ be the bundle of frames in the tangent bundle TX. Then FX is a principal $GL(d-1, \mathbb{R})$-bundle over X, and the fibre product $FX \times_X FX$ is a principal $GL(d-1, \mathbb{R}) \times GL(d-1, \mathbb{R})$-bundle over X; its total space is a real affine variety. We leave the reader the task to apply Theorem 4.1 and provide some examples of sums of squares decompositions of sparse polynomials defined on $FX \times_X FX$.

In general, if $f_{1,2} : E \longrightarrow Y$ are two identical principal bundles with real-algebraic affine global spaces and with structural algebraic groups G, then the fibre product $Z = E_1 \times_Y E_2$ consists of

$$Z = \{(gx, x);\ x \in E, g \in G\}.$$

The space of sparse elements corresponds to $(\mathbb{R}[E] + \mathbb{R}[E])|_Z$, whence it can be identified to the set of polynomial functions defined on $E \times G$:

$$p(x) + q(gx), \quad p, q \in \mathbb{R}[E],\ x \in E,\ g \in G.$$

For example, let n be a positive integer, and consider the unit circle $S^1 \subset \mathbb{R}^2$, with complex coordinate $z \in S^1,\ |z| = 1$, and the unramified finite covering

$$S^1 \longrightarrow S^1, \quad z \mapsto z^n.$$

Then the fibre product of two copies of the same S^1-principal bundle has the total space

$$Z = S^1 \times_{S^1} S^1 = S^1 \times \mathbb{Z}_n.$$

The space of sparse (trigonometric in this case) polynomials is

$$V = \{p(z, \overline{z}) + q(\epsilon^k z, \epsilon^{-k}\overline{z});\ |z| = 1, 0 \leq k \leq n-1\},$$

where $\epsilon = exp(\frac{2\pi i}{n})$ and p, q are real valued (i.e. hermitian) polynomials:

$$p(z, \overline{z}) = \overline{p(z, \overline{z})},\ q(z, \overline{z}) = \overline{q(z, \overline{z})}.$$

Our main result yields.

Proposition 3. *Let $p(z,\overline{z}), q(z,\overline{z})$ be real valued trigonometric polynomials. If*

$$(|z|=1, 0\le k\le n-1) \Rightarrow \; p(z,\overline{z}) + q(\epsilon^k z, \epsilon^{-k}\overline{z}) > 0,$$

then there are finitely many complex analytic polynomials $s_i(z), t_j(z)$, such that

$$(|z|=1, 0\le k\le n-1) \Rightarrow p(z,\overline{z})+q(\epsilon^k z, \epsilon^{-k}\overline{z}) = \sum_i |s_i(z)|^2 + \sum_j |t_j(\epsilon^k z)|^2.$$

For the proof we simply remark that for every real valued trigonometric polynomial

$$h(z,\overline{z}) = \sum_{j=-k}^{k} c_k z^k = z^{-k}\sum_{j=0}^{2k} c_k z^{k+j},$$

so that

$$h(z,\overline{z})^2 = |\sum_{j=0}^{2k} c_k z^{k+j}|^2, \quad |z|=1.$$

6.6. **The Hopf bundle.** In the same spirit as in the last example we consider below the three dimensional sphere

$$S^3 = \{z\in\mathbb{C}^2;\ |z|^2 = |z_1|^2+|z_2|^2 = 1\}$$

and the (twisted) Hopf fibrations on it

$$H_k = \{(z,\lambda z^k), z\in S^3,\ |\lambda|=1\},$$

where k is a non-negative integer and

$$z^k = (z_1^k, z_2^k).$$

It is clear that H_k are real algebraic, affine varieties, described in $\mathbb{C}^4$ by the equations

$$H_k\{(z,w)\in\mathbb{C}^2;\ |z|=1,\ w_1 z_2^k = w_2 z_1^k,\ |\frac{w_i}{z_i^k}|=1\}.$$

Thus the projection maps $H_k \longrightarrow S^3$ provide bundles in S^1-spheres.

The fibre product $Z = H_k \times_{S^3} H_\ell$ can be identified with the real affine subvariety of $\mathbb{C}^6$:

$$Z = \{(z,u,v);\ (z,u)\in H_k,\ (z,v)\in H_\ell\}.$$

The corresponding space of sparse hermitian (real valued) polynomials is

$$V = \{p(z,u;\overline{z},\overline{u}) + q(z,v;\overline{z},\overline{v});\ (z,u) \in H_k,\ (z,v) \in H_\ell\}.$$

Remark the free action of the group $S^1 \times S^1$ on the variables (u,v).

Thus, we are led to the following conclusion.

Proposition 4. *Assume that the hermitian polynomials p,q satisfy*

$$[(z,u) \in H_k,\ (z,v) \in H_\ell] \ \Rightarrow p(z,u;\overline{z},\overline{u}) + q(z,v;\overline{z},\overline{v}) > 0.$$

Then there are finitely many hermitian polynomials s_i, t_j, such that

$$[(z,u) \in H_k,\ (z,v) \in H_\ell] \ \Rightarrow p(z,u;\overline{z},\overline{u}) + q(z,v;\overline{z},\overline{v}) =$$

$$\sum_i |s_i(z,u;\overline{z},\overline{u})|^2 + \sum_j |t_j(z,v;\overline{z},\overline{v})|^2.$$

6.7. **Cylindrical sets with the (‡) property.** Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $y \in \mathbb{R}$. We are concerned with cylindrical sets with basis in $\mathbb{R}^n$ of the form $K(Q_1') \times \mathbb{R}$, where $Q_1' \subset \mathbb{R}[x]$ is an archimedean preorder. When regarding $Q_1' \subset \mathbb{R}[x,y]$, this preorder ceases to have the archimedean property, but it satisfies condition (‡), see [17].

We would like to analyze below the fibre product of two such structure, and compare our sparse polynomial decomposition, to the standard full algebra one. Specifically, consider another system of variables $z = (z_1, ..., z_m)$ and an archimedean preorder $Q_2' \subset \mathbb{R}[z]$. Let

$$f_1 : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}, \quad f_2 : \mathbb{R}^m \times \mathbb{R} \longrightarrow \mathbb{R}$$

be the projection maps. The fibre product of the two arrows is the space $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$ with coordinates (x,y,z). Assume that the positivity sets $K_{1,2} = K(Q_{1,2}')$ have non-empty interior in $\mathbb{R}^n$, respectively $\mathbb{R}^m$. Then the projection maps $f_i : K_i \times \mathbb{R} \longrightarrow \mathbb{R}$ are obviously proper.

Let $Q_{1,2}$ be the preorders generated by $Q_{1,2}'$ in the algebras $\mathbb{R}[x,y]$, respectively $\mathbb{R}[y,z]$.

In virtue of Lemma 5 the convex cone

$$Q_1 + Q_2 \subset \mathbb{R}[x,y] + \mathbb{R}[y,z]$$

has property (‡) with respect to the same sparse subspace. Its positivity set is

$$K = K(Q_1 + Q_2) = K_1 \times \mathbb{R} \times K_2,$$

hence again a cylinder with generatrix parallel to the y direction. Thus we conclude that for every $f \in \mathbb{R}[x, y] + \mathbb{R}[y, z]$ which non-negative on K there exists an element $q \in Q_1 + Q_2$ such that for all $\epsilon > 0$ we have $f + \epsilon q \in Q_1 + Q_2$.

On the other hand, property (‡) applied to the preorder Q generated by Q_1, Q_2 in $\mathbb{R}[x, y, z]$ gives for the same element a $q \in Q$, such that $f + \epsilon q \in Q$.

In both cases one can show as in [17] that $q = q(y)$ can be chosen to be an universal polynomial depending only on y and the degree of f.

To give a particular example, let us choose preorders generated by a single polynomial:

$$Q_1' = \Sigma\mathbb{R}[x]^2 + h_1(x)\Sigma\mathbb{R}[x]^2,$$

$$Q_2' = \Sigma\mathbb{R}[z]^2 + h_2(z)\Sigma\mathbb{R}[z]^2.$$

Then

$$Q_1 = \Sigma\mathbb{R}[x, y]^2 + h_1(x)\Sigma\mathbb{R}[x, y]^2,$$

and

$$Q_2 = \Sigma\mathbb{R}[y, z]^2 + h_2(z)\Sigma\mathbb{R}[y, z]^2.$$

Thus, if

$$f \in \mathbb{R}[x, y] + \mathbb{R}[y, z], \quad f|_K \geq 0,$$

there exists $q \in \mathbb{R}[x, y] + \mathbb{R}[y, z]$, such that for all $\epsilon > 0$,

$$f + \epsilon q \in \Sigma\mathbb{R}[x, y]^2 + h_1(x)\Sigma\mathbb{R}[x, y]^2 + \Sigma\mathbb{R}[y, z]^2 + h_2(z)\Sigma\mathbb{R}[y, z]^2.$$

On the other hand, the standard Positivstellensatz implied by property (‡) in the set of all variables, only gives:

$$f + \epsilon q \in \Sigma\mathbb{R}[x, y, z]^2 + h_1(x)\Sigma\mathbb{R}[x, y, z]^2 + h_2(z)\Sigma\mathbb{R}[x, y, z]^2 + h_1(x)h_2(z)\Sigma\mathbb{R}[x, y, z]^2.$$

6.8. **Approximating holomorphic functions of an infinity of variables.** The problem of approximating on convex sets holomorphic functions of infinitely many variables by polynomials is rather delicate, and not fully solved, see for instance [22]. We provide a simple application of Proposition 2.

Let $x = \{x_1, x_2, ...\}$ be a countable system of variables, seen as coordinates in the real Hilbert space $H = l^2(\mathbb{N})$. Let $B_1(H) = \{x \in H; \|x\|^2 = x_1^2 + x_2^2 + ... \leq 1\}$ be the closed unit ball. Endowed with the weak topology, we can regard $B_1(H) = \text{proj.lim}(B_1(V), r_{V,W})$, where V runs over all finite dimensional subspaces of H, and $r_{V,W} : B_1(W) \longrightarrow B_1(V)$ is the orthogonal projection map, whenever $V \subset W$. By passing to an increasing subsequence of subspaces, say

$$V_n = \{(x_1, x_2, ., , , x_n, 0, 0, ...); x_i \in \mathbb{R}\},$$

we still have the topological identification

$$B_1(H) = \text{proj.lim}(B_1(V_n), r_{V_n,V_{n+1}}).$$

Let $\Sigma\mathbb{R}[x_1, ..., x_n]^2$ denote, as before, the convex cone of sums of squares of polynomials, in the respective number of variables.

Let $F : B_1(H) \longrightarrow \mathbb{R}$ be the germ of a holomorphic function (see [22] for the precise definition). In particular F is a weak limit of a sequence of polynomials in finitely many (but increasing in number) of variables. The conditions of Proposition 2 are met, and we can state the following result.

Proposition 5. *Let $F : B_1(H) \longrightarrow \mathbb{R}$ be the germ of a holomorphic, non-negative function. Then there are polynomials $p_n \in \Sigma\mathbb{R}[x_1, ..., x_n]^2 + (1 - x_1^2 - ... - x_n^2)\Sigma\mathbb{R}[x_1, ..., x_n]^2$, such that $F = w - \lim_n p_n$, more precisely the convergence is uniform on every cylinder over $B_1(V_k)$, $k \geq 1$.*

As a matter of fact, we can prove a more refined statement, which will go beyond any standard method of approximation by power series. Namely, assume that the non-negative function F and the set $B_1(H)$ in the statement have some sparsity structure, subject to the condition $\mathcal{R}$. For instance, assume that we replace $B_1(H)$ by the closed set in H:

$$K = \{x;\ x_{2k}^2 + x_{2k+1}^2 + x_{2k+2}^2 \leq 3^{-k},\ k \geq 1\}.$$

Fix a positive integer N. Then the finite projective limit over $k \leq N$ of real affine varieties $Spec\mathbb{R}[x_{2k}, x_{2k+1}, x_{2k+2}]$ and quadratic modules

$$Q_k = \Sigma\mathbb{R}[x_{2k}, x_{2k+1}, x_{2k+2}]^2 + (3^{-k} - x_{2k}^2 - x_{2k+1}^2 - x_{2k+2}^2)\Sigma\mathbb{R}[x_{2k}, x_{2k+1}, x_{2k+2}]^2$$

satisfy the condition in our main result. Denote $K_N = \text{proj.lim}_{k=1}^N P(Q_k)$. In its turn, $K = \text{proj.lim} K_N$.

Corollary 6.2. *In the above conditions, assume that in the germ of holomorphic function $F : K \longrightarrow \mathbb{R}$ only monomials of the form $x_{2k}^p x_{2k+1}^q x_{2k+2}^r$, $k \geq 1$, appear. If F is non-negative on K, then there are polynomials $p_k \in Q_k$, such that $F = w - \lim \sum_k p_k$.*

The reader will easily construct similar examples based on the above pattern.

6.9. **Countably many constraints.** Every compact subset $K \subset \mathbb{R}^d$ can be written as an intersection of countable many semi-algebraic sets:

$$K = \{x \in \mathbb{R}^d;\ q_i(x) \geq 0,\ i \geq 1\}.$$

Let $\{I_1, ..., I_k\}$ be a covering of the set of indices $\{1, ..., d\}$ satisfying the running intersection property, as in Kojima-Lasserre Theorem. Assume that the defining functions $q_i, i \geq 1$, are subject to this sparsity pattern (that is q_i contains only monomials in a set of variables $I_{j(i)}$ depending on i), and that they define archimedean quadratic modules. In particular

$$Q_m = \sum_j \Sigma\mathbb{R}[x(I_j)]^2 + q_1\Sigma\mathbb{R}[x(I_{j(1)})]^2 + ... + q_m\Sigma\mathbb{R}[x(I_{j(m)})]^2$$

are convex cones for all m; they possess the (SMP) by our main result, and have compact positivity sets. Thus, Proposition 2 applies, and we obtain the following conclusion.

Proposition 6. *With the above assumptions, let $p \in \sum_j \mathbb{R}[x(I_j)]$ be a sparse polynomial which is non-negative on the set K. Then there is a sequence of polynomials $q_m \in Q_m$ which converges uniformly to p on K.*

Note that a Stone-Weierstrass argument cannot be used in general to prove such a statement, due to the fact that sparse polynomials do not form an algebra.

References

[1] C. Bayer, J. Teichman, *The proof of Tchakaloff's Theorem*, Proc. Amer. Math. Soc. **134**(2006), 3035-3040.

[2] J.R.S. Blair, B. Peyton, *An introduction to chordal graphs and clique trees,* in vol. " Graph theory and sparse matrix computation",pp. 1–29, IMA Vol. Math. Appl.**56**, Springer, New York, 1993.

[3] S. Bochner, *Harmonic analysis and the theory of probability*, Univ. California Press, Berkeley, 1955.

[4] N. Bourbaki, *Intégration*, Chapitre IX, Hermann, Paris, 1969.

[5] J. Cimpric, S. Kuhlmann, C. Scheiderer, *Sums of squares and invariant moment problems in equivariant situations*, submitted for publication.

[6] W. Feller, *An introduction to probability theory and its applications.* Vol. II. Second edition, John Wiley & Sons, Inc., New York-London-Sydney, 1971.

[7] M. Fukuda, M. Kojima, K. Murota, K. Nakata, *Exploiting sparsity in semidefinite programming via matrix completion. I. General framework,* SIAM J. Optim. **11** (2000/01), no. 3, 647–674.

[8] D. Grimm, T. Netzer, M. Schweighofer, *A note on the representation of positive polynomials with structured sparsity*, prerpint 2006.

[9] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. **52**. Springer-Verlag, New York-Heidelberg, 1977

[10] J.W. Helton, M. Putinar, *Positive polynomials in scalar and matrix variables, the spectral theorem, and optimization*, in vol. *Structured Matrices and Dilations. A Volume Dedicated to the Memory of Tiberiu Constantinescu* (M. Bakonyi et al. eds.), Theta, Bucharest, to appear.

[11] D. Husemoller, *Fibre bundles.* Third edition. Graduate Texts in Mathematics, **20**. Springer-Verlag, New York, 1994.

[12] C.R. Johnson, *Matrix completion problems: a survey.* Matrix theory and applications (Phoenix, AZ, 1989),pp. 171–198, Proc. Sympos. Appl. Math., 40, Amer. Math. Soc., Providence, RI, 1990.

[13] S. Kim, M. Kojima, H. Waki, *Generalized Lagrangian duals and sums of squares relaxations of sparse polynomial optimization problems*, SIAM J. Optim. **15** (2005), no. 3, 697–719.

[14] G. Köthe, *Topological Vector Spaces. I.* Springer, Berlin (1969).

[15] M. Kojima, S. Kim, H. Waki, *Sparsity in sums of squares of polynomials*, Math. Program. **103** (2005), no. 1, Ser. A, 45–62.

[16] M. Kojima, M. Muramatsu, *A note on sparse SOS and SDP relaxations for polynomial optimization problem over symmetric cones*, Res. Report B-421, Math. Dept. Tokyo Inst. Techn. (2006).

[17] S. Kuhlmann, M. Marshall, *Positivity, sums of squares, and the multi-dimensional moment problem I*, Transactions Amer. Math. Soc. **354** (2002), 4285-4301

[18] S. Kuhlmann, M. Marshall, N. Schwartz *Positivity, sums of squares, and the multi-dimensional moment problem II*, Adv. Geom. **5** (2005), 583-607.

[19] J.B. Lasserre, *Global optimization with polynomials and the problem of moments*, SIAM J. Optim. **11** (2000/01), no. 3, 796–817

[20] J.B. Lasserre, *Convergent semidefinite relaxations in polynomial optimization with sparsity*, SIAM J. Optim. **17**(2006), 796-817.

[21] J. B. Lasserre, *A Positivstellensatz which preserves the coupling pattern of variables* ArXiv: math.AC/0609529, 11pp.

[22] L. Lempert, *Approximation de fonctions holomorphes d'un nombre infini de variables*, Annales de l'Institut Fourier, **49** no. 4 (1999), 1293-1304.

[23] K. Nakata, K. Fujisawa, M. Fukuda, M. Kojima, K. Murota, *Exploiting sparsity in semidefinite programming via matrix completion. II,* Math. Program. **95** (2003), no. 2, Ser. B, 303–327.

[24] J. Nie, J. Demmel, *Sparse SOS relaxations for minimizing functions that are summation of small polynomials*, preprint, http://arxiv.org.abs.math.OC/0606476

[25] V. Powers, C. Scheiderer, *The moment problem for non-compact semialgebraic sets*, Adv. Geom. **1** (2001), 71-88.

[26] A. Prestel, C. Delzell, *Positive polynomials*, Springer, Berlin, 2001.

[27] M. Putinar, *Positive polynomials on compact semi-algebraic sets*, Indiana Univ. Math. J. **42**, (1993) 969-984

[28] M. Putinar, C. Scheiderer, *Multivariate moment problems: geometry and indeterminateness*, Ann. Scuola Norm. Sup. Pisa Cl. Sci (5), vol. V(2006), 137-157.

[29] B. Reznick, *Sums of Even Powers of Real Linear Forms*, Memoirs Amer. Math. Soc. vol. **96** No. 463, Amer. Math. Soc., Providence, R.I., 1992.

[30] R. T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, Princeton, 1972.

[31] K. Schmüdgen, *The K-moment problem for compact semi-algebraic sets*, Math. Ann. **289** (1991) 203-206.

[32] K. Schmüdgen, *On the moment problem of closed semi-algebraic sets*, J. reine angew. Math. **558**(2003), 225-234.

[33] H. Waki, S. Kim, M. Kojima, M. Muramatsu, *Sums of squares and semidefinite programming relaxations for polynomial optimization problems with structured sparsity*, SIAM J. Optim. **17**(2006), 218-242.

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