Notes on generalized lemniscates

Mihai Putinar

Abstract. A series of analytic and geometric features of generalized lemniscates are presented from an elementary and unifying point of view. A novel interplay between matrix theory and elementary geometry of planar algebraic curves is derived, with a variety of applications, ranging from a classical Fredholm eigenvalue problem and Hardy space estimates to a root separation algorithm.

1. Introduction

The object of study in this article is the level set, with respect to a Hilbert space norm, of the resolvent of a matrix localized at a vector. We call these sets *generalized lemniscates* in analogy with the classical lemniscates, that is level sets of the modulus of a polynomial. The latter class of domains is well known for its applications to approximation theory and potential theory, see for instance [21] and [29].

The origin of this study goes back to some determinantal functions related to the spectral theory of hyponormal operators. More specifically, if $T \in L(H)$ is a linear bounded operator acting on the Hilbert space H and the commutator $[T^*, T] = \xi \otimes \xi$ is non-negative and rank-one, then the infinite determinant:

$$\det[(T^* - \overline{z})^{-1}(T - z)(T^* - \overline{z})(T - z)^{-1}] =$$
$$1 - \|(T^* - \overline{z})^{-1}\xi\|^2$$

was instrumental in understanding the fine structure of these operators. Its properties were first investigated from the point of view of perturbation

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theory of symmetric operators and of scattering theory, see [24], [6]; for more references and later related works see [23] and [27].

We have remarked in [26] that all rank-one self-commutator operators T as above, subject to the additional condition

$$\dim \bigvee_{k=0}^{\infty} T^{*}\xi < \infty,$$

are in a natural correspondence to Aharonov and Shapiro's quadrature domains ([2]). In particular, the spectrum of these finite type operators is precisely given by the rational equation:

$$1 - \|(A - \overline{z})^{-1}\xi\|^{2} < 0,$$

where A is the finite matrix obtained by compressing T^* to $\bigvee_{k=0}^{\infty} T^{*}\xi$. This is a generalized lemniscate, in the terminology adopted below. This very correspondence between planar algebraic curves and matrices proved to be fruitful for better understanding the nature of the defining equation of the boundary of a quadrature domain (already investigated by Gustafsson [13]).

An effective exact reconstruction algorithm of a quadrature domains from a part of its moments was also derived from the same observation, [11]. Although closely related to generalized lemniscates, we will not expand here these ideas.

The class of generalized lemniscates has emerged from such concepts and computations. We have tried below to simplify the access to these planar domains and to make it independent of any sophisticated theory of semi-normal operators. Most of the operator theory and approximation theory aspects as well as important connections to extremal moment problems are left aside. They are partially explained in the recent survey article [27], or in the more technical papers cited there.

One of the aims of this essay is to connect in simple terms a variety of ideas of linear algebra, realization theory of linear systems, algebraic geometry and some classical analysis on planar domains. We are well aware that this is only a first step. The material below was freely borrowed and compiled into another form from a series of recent articles published in the last five years: [5], [17], [25], [28]. This was done with the hope that the entire is more than its parts.

The contents is self-explanatory. Appropriate comments are added in the body of each section:

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2. Realization theory

In this section we link the specific form of the algebraic equation of a generalized lemniscate to a Hilbert space realization of it. This is in accord to the well known matrix realization of transfer functions in linear systems theory.

Let

$$Q(z,\overline{z}) = \sum_{j,k=0}^{d} \alpha_{jk} z^j \overline{z}^k,$$

be a Hermitian polynomial in $(z,\overline{z})$, that is $\alpha_{jk} = \overline{\alpha_{kj}}$, $1 \le j,k \le d$. We will assume that the leading coefficient is non-zero, and normalized:

$$\alpha_{dd} = 1,$$

and we denote

$$P(z) = \sum_{j=0}^{d} \alpha_{jd} z^j.$$

Then

$$|P(z)|^2 - Q(z,\overline{z}) = \sum_{j,k=0}^{d} [\alpha_{jd}\alpha_{dk} - \alpha_{jk}] z^j \overline{z}^k.$$

The following result, proved in [16], can be taken as a starting point for our discussion.

Theorem 2.1. *The following conditions are equivalent:*

a) The matrix $A(\alpha) = (\alpha_{jd}\alpha_{dk} - \alpha_{jk})_{j,k=0}^{d-1}$ *is strictly positive definite;*

b) There exists a linear transformation A *of* $\mathbf{C}^d$ *with a cyclic vector* ξ *so that* $P(A) = 0$ *and*

$$\frac{Q(z,\overline{z})}{|P(z)|^2} = 1 - \|(A-z)^{-1}\xi\|^2; \tag{2.1}$$

c) There exist polynomials $Q_k(z)$ *of degree* k *(exactly),* $0 \leq k < d$*, with the property*

$$Q(z,\overline{z}) = |P(z)|^2 - \sum_{k=0}^{d-1} |Q_k(z)|^2. \tag{2.2}$$

In c), the $Q_k^{\prime}s$ *are uniquely determined if the leading coefficients are required to be positive.*

Proof. Below we simply sketch the main steps of the proof. More details are included in [16].

a) $\Rightarrow$ **b)**. Assume that $A(\alpha)$ is positive definite. Then there exist linearly independent vectors $v_k \in \mathbf{C}^d$, $0 \leq k < d$, satisfying:

$$\langle v_j, v_k \rangle = \alpha_{jd}\alpha_{dk} - \alpha_{jk},$$

and consequently

$$|P(z)|^2 - Q(z,\overline{z}) = \|V(z)\|^2, \tag{2.3}$$

where $V(z) = \sum_{j=0}^{d-1} v_j z^j$ is a vector-valued polynomial.

It follows that $R : \mathbf{P}_1 \to \mathbf{P}_d$, defined in terms of homogeneous coordinates in $\mathbf{P}_d$ by $R(z) = (P(z) : V(z))$, is a rational map of degree d such that the image of R spans $\mathbf{P}_d$. Elementary arguments of linear algebra imply then the existence of a matrix $A \in L(\mathbf{C}^d)$ with minimal polyonmial $P(z)$ and cyclic vector ξ, such that:

$$V(z) = P(z)(A-z)^{-1}\xi.$$

This proves assertion *b*).

b) $\Rightarrow$ **c)**. To achieve the decomposition (2.2) we orthonormalize the vectors $\xi, A\xi, \ldots, A^{d-1}\xi$:

$$e_0 = \frac{\xi}{\|\xi\|},$$

$$e_1 = \frac{A\xi - \langle A\xi, e_0\rangle e_0}{\|\ldots\|},$$

$$e_2 = \frac{A^2\xi - \langle A^2\xi, e_1\rangle e_1 - \langle A^2\xi, e_0\rangle e_0}{\|\ldots\|},$$

etc. Equivalently,

$$\xi = \|\xi\| e_0 = c_0 e_0 \quad (c_0 > 0),$$
$$A\xi = c_1 e_1 + \langle A\xi, e_0\rangle e_0 \quad (c_1 > 0),$$
$$A^2\xi = c_2 e_2 + \langle A^2\xi, e_1\rangle e_1 + \dots \quad (c_2 > 0),$$

and so on. By rearranging the terms we obtain:

$$\begin{aligned}
-P(z)(A-z)^{-1}\xi &= (P(A)-P(z))(A-z)^{-1}\xi = T_0(z)A^{d-1}\xi + \dots + T_{d-1}(z)\xi = \\
&= T_0(z)(c_{d-1}e_{d-1} + \langle A^{d-1}\xi, e_{d-2}\rangle e_{d-2} + \dots) + \\
&+ T_1(z)(c_{d-2}e_{d-2} + \langle A^{d-2}\xi, e_{d-3}\rangle e_{d-3} + \dots) + \dots + T_{d-1}(z)c_0 e_0 = \\
&= c_{d-1}T_0(z)e_{d-1} + (c_{d-2}T_1(z) + \langle A^{d-1}\xi, e_{d-2}\rangle T_0(z))e_{d-2} + \dots + \\
&+ (c_0 T_{d-1}(z) + \langle A\xi, e_0\rangle T_{d-2}(z) + \dots)e_0 = \\
&= Q_0(z)e_{d-1} + Q_1(z)e_{d-2} + \dots + Q_{d-1}(z)e_0,
\end{aligned}$$

where

$$Q_k(z) = c_{d-1-k}T_k(z) + O(z^{k-1}).$$

Hence $Q_k(z)$ is a polynomial of degree k with leading coefficient $c_{d-1-k} > 0$, and (2.2) now follows by inserting the above expression for $P(z)(A-z)^{-1}\xi$ into (2.1) and using that the e_j are orthonormal.

c) $\Rightarrow$ **a)**. If assertion c) is assumed to be true, then the vector-valued polynomial

$$V(z) = (Q_0(z), Q_1(z), \dots, Q_{d-1}(z))$$

satisfies (2.3). Expanding $V(z)$ along increasing powers of z gives $V(z) = \sum_{j=0}^{d-1} v_j z^j$ where the v_j are linearly independent vectors. Then (2.3) and show that $A(\alpha)$ is a strictly positive Gram matrix (associated to the vectors v_j). Hence $A(\alpha)$ is strictly positive definite, proving a).

It remains to prove the uniqueness of the decomposition (2.2). For this we observe that there exists a simple algorithm of finding the polynomials Q_k. Indeed, first observe that the coefficient of $\overline{z}^d$ in $Q(z,\overline{z})$ is $P(z)$. Hence the polynomial $F_{d-1}(z,\overline{z}) = |P(z)|^2 - Q(z,\overline{z})$ has degree $d-1$ in each variable. By assumption the coefficient γ_1 of $z^{d-1}\overline{z}^{d-1}$ in F_{d-1} is positive, so that:

$$F_{d-1}(z,\overline{z}) = {\gamma_1}^{1/2}\overline{z}^{d-1}Q_{d-1}(z) + O(z^{d-1}, \overline{z}^{d-2}).$$

Therefore the polynomial $Q_{d-1}(z)$ is determined by $F_{d-1}(z,\overline{z})$.

Proceeding by descending recurrence in k, $(k < d-1)$ we are led to the polynomial

$$F_k(z,\overline{z}) = F_{k+1}(z,\overline{z}) - |Q_{k+1}(z)|^2$$

which has as leading term a positive constant γ_k times $z^k\overline{z}^k$. Then necessarily

$$F_k(z,\overline{z}) = {\gamma_k}^{1/2}\overline{z}^k Q_k(z) + O(z^k, \overline{z}^{k-1}).$$

Thus $Q_k(z)$ is determined by $F_k(z,\overline{z})$. And so on until we end by setting $F_0(z,\overline{z}) = \gamma_0 = |Q_0(z,\overline{z})|^2 > 0$.

□

Definition 2.2. A *generalized lemniscate* is a bounded open set Ω of the complex plane, given by the equation:

$$\Omega = \{z \in \mathbf{C};\ \|(A-z)^{-1}\xi\| > 1\},$$

where $A \in L(\mathbf{C}^d)$ is a linear transformation and ξ is a cyclic vector of it, or equivalently:

$$\Omega = \{z \in \mathbf{C};\ |P(z)|^2 - \sum_{k=0}^{d-1} |Q_k(z)|^2 < 0\},$$

with polynomials P, Q_k subject to the degree conditions: $\deg P = d,\ \deg Q_k = k,\ 0 \leq k \leq d-1$.

Henceforth we call the pair (A,ξ) the *linear data* of a generalized lemniscate.

At this point we can easily make the link to the theory of *determinantal curves* due to Moshe Livsič, Kravitsky, Vinnikov and their school. Specifically, starting with a matrix $A \in L(\mathbf{C}^d)$ as above and a cyclic vector ξ of it, we can produce a linear pencil of matrices having the determinant equal to the polynomial Q above. Indeed,

$$\begin{vmatrix} \xi\langle\cdot,\xi\rangle & A-z \\ A^*-\overline{z} & I \end{vmatrix} = \begin{vmatrix} \xi\langle\cdot,\xi\rangle - (A-z)(A^*-\overline{z}) & A-z \\ 0 & I \end{vmatrix} =$$

$$\det[\xi\langle\cdot,\xi\rangle - (A-z)(A^*-\overline{z})] =$$

$$|\det(A-z)|^2 \det[(A-z)^{-1}\xi\langle\cdot,(A-z)^{-1}\xi\rangle - I] =$$

$$-|\det(A-z)|^2[1 - \|(A-z)^{-1}\xi\|^2.$$

Thus we can state

Proposition 2.3. *A generalized lemniscate Ω with linear data (A,ξ) is given by the determinantal equation:*

$$\Omega = \{z \in \mathbf{C}; \begin{vmatrix} \xi\langle\cdot,\xi\rangle & A-z \\ A^*-\overline{z} & I \end{vmatrix} > 0\}. \tag{2.4}$$

We refer to [20], [34], [35] and the monograph [22] for the theory of determinantal curves. Again, we do not expand here the predictable implications of formula (2.4). Some of them have been considered by Alex. Shapiro [32].

3. The rational embedding

The realization of a generalized lemniscate as the level set of the resolvent of a matrix has immediate geometric interpretations and consequences. One of them is the derivation of a canonical rational embedding in an affine, or projective complex space. Full proofs and a more detailed analysis of these aspects are contained in [16] Sections 4 and 5.

Henceforth we denote by $\hat{\mathbf{C}}$ the Riemann sphere (that is the compactification of the complex plane by one point at infinity). Equivalently, this is the projective space of dimension one: $\hat{\mathbf{C}} = \mathbf{P}_1(\mathbf{C})$. The projective space of dimension d will be denoted by $\mathbf{P}_d(\mathbf{C})$ or simply $\mathbf{P}_d$.

Let d be a positive integer, $d > 1$, let A be a linear transformation of $\mathbf{C}^d$, and assume that $\xi \in \mathbf{C}^d$ is a cyclic vector for A. Let us denote by :

$$R(z) = (A - z)^{-1}\xi, \quad z \in \mathbf{C} \setminus \sigma(A),$$

the resolvent of A, localized at the vector ξ.

Lemma 3.1. *The map $R : \hat{\mathbf{C}} \setminus \sigma(A) \longrightarrow \mathbf{C}^d$ is one to one and its range is a smooth complex curve.*

A complete proof is contained in [16] Lemma 4.1. The main idea is to consider the resolvent equation:

$$R(z) - R(w) = (z - w)(A - z)^{-1}(A - w)^{-1}\xi, \quad z, w \in \mathbf{C} \setminus \sigma(A).$$

Thus $R(z) - R(w) \neq 0$ for $z \neq w$. For the point at infinity we have $R(\infty) = 0 \neq R(z)$, for $z \in \mathbf{C} \setminus \sigma(A)$.

Moreover, the same resolvent equation shows that:

$$R'(z) = (A - z)^{-1}R(z) \neq 0,$$

and similarly for the point at infinity we obtain:

$$\frac{d}{dt}R(1/t) = -\lim_{t\to 0}[t^{-2}(A - t^{-1})^{-2}\xi] = -\xi \neq 0.$$

Actually we can pass to projective spaces and complete the above curve as follows. Let us denote by $(z_0 : z_1)$ the homogeneous coordinates in $\mathbf{P}_1$, and by $(u_0 : u_1 : \ldots : u_d)$ the homogeneous coordinates in $\mathbf{P}_d$. Let $z = z_1/z_0$ in the affine chart $z_0 \neq 0$ and $w = (u_1/u_0, \ldots, u_d/u_0)$ in the affine chart $u_0 \neq 0$.

Let $P(z) = det(A - z)$, so that $P(z)$ is a common denominator in the rational entries of the map $R(z)$. Let us define, as in the preceding section, the function:

$$q(z, A)\xi = P(z)R(z) = (P(z) - P(A))R(z),$$

and remark that $q(z, A)$ is a polynomial in z and A, of the form:

$$q(z, A) = -A^{d-1} + O(A^{d-2}).$$

Actually we need for later use a more precise form of the polynomial $q(z, A)$. We pause here to derive it by a series of elementary computations. We have

$$\frac{P(w) - P(z)}{w - z} = \sum_{k=0}^{d} \alpha_k \frac{w^k - z^k}{w - z} =$$

$$\sum_{k=0}^{d} \alpha_k \sum_{j=0}^{k-1} z^{k-j-1} w^j = \sum_{j=0}^{d-1} \left(\sum_{k=j+1}^{d} \alpha_k z^{k-j-1} \right) w^j =$$

$$T_0(z) w^{d-1} + T_1(z) w^{d-2} + \ldots + T_{d-1}(z),$$

where $\alpha_d = 1$ and

$$T_k(z) = \alpha_d z^k + \alpha_{d-1} z^{k-1} + \ldots + \alpha_{d-k+1} z + \alpha_{d-k}.$$

Note that $T_0(z) = 1$.

Therefore we obtain, as in the previous section:

$$-q(z, A) = T_0(z) A^{d-1} + T_1(z) A^{d-2} + \ldots + T_{d-1}(z). \tag{3.1}$$

Since ξ is a cyclic vector for A and $\dim \vee_{k=0}^{\infty} A^k \xi = d$, we infer that $q(z, A)\xi \neq 0$ for all $z \in \mathbf{C}$. In addition, for an eigenvalue λ of A (multiple or not), we have:

$$(A - \lambda) q(\lambda, A)\xi = P(\lambda)\xi = 0,$$

therefore $q(\lambda, A)\xi$ is a corresponding (non-trivial) eigenvector.

At this point we can define the completion of the map R as follows:

$$R(z_0 : z_1) = \begin{cases} (P(z_1/z_0) : q(z_1/z_0, A)\xi), & z_0 \neq 0, \\ (1 : 0 : \ldots : 0), & z_0 = 0. \end{cases} \tag{3.2}$$

By putting together these computations one obtains the following result.

Lemma 3.2. *The map $R : \mathbf{P}_1 \longrightarrow \mathbf{P}_d$ is a smooth embedding, that is, R is one to one and its image is a smooth projective curve.*

Note that $R(\mathbf{P}_1)$ is a smooth unirational curve of degree d in $\mathbf{P}_d$ and the rational map R has degree d. According to a classical result in algebraic geometry, $R(\mathbf{P}_1)$ is projectively isomorphic to the *rational normal curve* of degree d in $\mathbf{P}_d$ obtained as the range of the Veronese embedding

$$(z_0 : z_1) \longmapsto ({z_0}^d : {z_0}^{d-1} z_1 : \ldots : {z_1}^d).$$

See for details [12] pg. 178.

Actually the cyclicity condition on ξ can be dropped, because the resolvent $(A-z)^{-1}\xi$ has values in the cyclic subspace generated by ξ. Therefore, as a conclusion of these computations we can state the following result.

Theorem 3.3. *Let A be a linear transformation of $\mathbf{C}^d$ and let ξ be a non-zero vector of $\mathbf{C}^d$. Then the map $R(z)=(A-z)^{-1}\xi$ extends to a rational embedding:*

$$R : \mathbf{P}_1 \longrightarrow \mathbf{P}_d.$$

The range of R is contained in a linear subspace E of $\mathbf{P}_d$ of dimension equal to dim $\vee_{k=0}^{\infty} A^k\xi$ and the values $R(z)$ span E as a linear space.

Above, and throughout this note, by *embedding* we mean a (rational) map which separates the points and the directions at every point. In particular this implies that $R(\mathbf{P}_1)$ is a smooth rational curve.

Let us focus now on the geometry of the generalized lemniscate:

$$\Omega = \{z \in \mathbf{C}; \|(A-z)^{-1}\xi\| > 1\} \cup \sigma(A).$$

The singular points a in the boundary of the bounded domain Ω are given by the equation $\langle R'(a), R(a)\rangle = 0$. The proofs above show that $\|R'(a)\| \neq 0$, and on the other hand the Hessian $H(a)$ at a of the defining equation $\|R(z)\|^2 = 1$ is:

$$H(a) = \begin{pmatrix} \langle R'(a), R'(a)\rangle & \langle R''(a), R(a)\rangle \\ \langle R(a), R''(a)\rangle & \langle R'(a), R'(a)\rangle \end{pmatrix}.$$

In particular rank$H(a) \geq 1$, which shows that a is either an isolated point or a singular double point of $\partial\Omega$.

Our next aim is to study the reflection in the boundary of the domain Ω defined above. More precisely, for a point $s \in \mathbf{P}_1(\mathbf{C})$ we consider the multivalued Schwarz reflection in $\partial\Omega$ as the set of solutions solutions $z = r_1(s), \ldots, r_d(s)$ of the equation:

$$\langle R(s), R(z)\rangle = 1. \tag{3.3}$$

Proposition 3.4. *The multivalued reflection $s \mapsto (r_j(s))_{j=1}{}^d$ satisfies:*

a). All $r_j(s) \in \Omega,\ 1 \leq j \leq d$, for $s \in \mathbf{P}_1(\mathbf{C}) \setminus \overline{\Omega}$;

b). For an appropriate numbering of the r_j's, $r_1(s) = s$ and $r_j(s) \in \Omega,\ 2 \leq j \leq d$, for $s \in \partial\Omega$.

Proof. Indeed, $\|R(s)\| < 1$ whenever s does not belong to $\overline{\Omega}$. Therefore $\|R(z)\| > 1$ for every solution z of the equation (3.3). For $s \in \partial\Omega$ we obtain $\|R(s)\| = 1$, hence one solution of (16), say r_1, satisfies $r_1(s) = s$ and all other solutions z satisfy necessarily $\|R(z)\| > 1$. □

A rigidity result of the above rational embedding in the complement of the sphere, compatible to the reflections in the boundaries, is discussed in detail in [16] Section 5.

This is an appropriate moment to recall the definition of a quadrature domain, in the sense of Aharonov and Shapiro [2].

Definition 3.5. A bounded planar open set Ω is a quadrature domain if it is real algebraic and the function $z \mapsto \overline{z}$ extends continuously from $z \in \partial\Omega$ to a meromorphic function in Ω.

This means, in line with Proposition 3.4, that one determination, say $S_1(z)$, of the Schwarz reflection satisfies $S_1(z) = \overline{z},\ z \in \partial\Omega$, and it does not have ramification points inside Ω. Necessarily, it will have d poles $a_1, \ldots, a_d,\ d \geq 1$, there. The number d is called the order of a quadrature domain and the terminology comes from the simple observation that, in this case there are d weights $c_1, c_2, \ldots, c_d$ with the property:

$$\int_\Omega f(z) d\text{Area}(z) = c_1 f(a_1) + \ldots + c_d f(a_d), \tag{3.4}$$

for every integrable analytic function f in Ω, see [2] and [33]. If multiple poles of $S_1(z)$ occur, then higher order derivatives of f evaluated there must be considered, correspondingly. As a matter of fact, the existence of the quadrature formula (3.4), valid for *all* integrable analytic functions is equivalent to the above definition of a quadrature domain. Since their discovery thirty years ago, [2] and [31], these domains have revealed a series of remarkable features, related to phenomena of function and potential theory, fluid mechanics, moment problems and partial differential equations, [33].

The case $d = 1$ corresponds to a disk. By abuse of language we allow non-connected sets in the above definition. Thus, a disjoint union of disks is also a quadrature domain.

Quadrature domains are relevant for this survey because of the following result (which as a matter of fact was the origin of the whole project).

Theorem 3.6. *A quadrature domain of order d is a generalized lemniscate of degree (d, d).*

The original proof of this theorem was based on non-trivial results of the theory of semi-normal operators ([26]). An elementary way to prove it was recently described in [17].

4. Fredholm eigenvalues

In this section we use a simple geometric feature of the multivalued Schwarz reflection in the boundary of a generalized lemniscate and prove that a classical problem in potential theory does not have non-trivial solutions on this class of domains. It is worth mentioning that the similar picture on classical lemniscates is quite different.

Let Ω be a bounded, simply connected domain of the complex plane and assume that the boundary of Ω is smooth.

Let $u \in C(\partial\Omega)$ be a continuous function. The *double layer potential* of u, with respect to Ω is the harmonic function:

$$D(u)(z) = \frac{1}{2\pi} \int_{\partial\Omega} u(\zeta) d \arg(\zeta - z).$$

An elementary computation shows that:

$$D(u)(z) = \Re[\frac{1}{2\pi i} \int_{\partial\Omega} \frac{u(\zeta)}{\zeta - z} d\zeta] = \int_{\partial\Omega} u(\zeta) \Re[\frac{d\zeta}{2\pi i(\zeta - z)}].$$

Whenever z belongs to Ω, respectively to its exterior, we mark the function D by an index $D_i(z)$ respectively $D_e(z)$. It is known that D_i is a continuous function on the closure $\overline{\Omega}$ and that D_e is continuous on the Riemann sphere $\hat{\mathbf{C}}$ minus Ω. Moreover, at each boundary point $\sigma \in \partial\Omega$ we have representations:

$$D_i(u)(\sigma) = \frac{1}{2}u(\sigma) + \frac{1}{2}K_\Omega(u)(\sigma),$$
$$D_e(u)(\sigma) = -\frac{1}{2}u(\sigma) + \frac{1}{2}K_\Omega(u)(\sigma).$$

Thus, the jumping formula:

$$D_i(u)(\sigma) - D_e(u)(\sigma) = u(\sigma), \quad \sigma \in \partial\Omega,$$

holds. Remark also that for the constant function $u = \mathbf{1}$ we have $D_i(\mathbf{1}) = \mathbf{1}$ and $D_e(\mathbf{1}) = 0$, hence $K_\Omega(\mathbf{1}) = \mathbf{1}$.

The linear continuous transformation $K_\Omega : C(\partial\Omega) \longrightarrow C(\partial\Omega)$ is the classical Neumann-Poincaré singular integral operator (in two real variables). In general this operator has better smoothness properties than the Hilbert transform.

Carl Neumann's approach to the Dirichlet problem $\Delta f = 0$ in Ω, $f|_{\partial\Omega} = u$, was essentially the following:

Solve the equation $1/2(I + K_\Omega)v = u$ and then represent $f = D_i(v)$ as a double layer potential v.

Thus, knowing that the operator $I + K_\Omega : C(\partial\Omega) \longrightarrow C(\partial\Omega)$ is invertible solves the Dirichlet problem for an arbitrary (originally convex) domain. Later this idea was applied by Poincaré, Fredholm, Carleman to more general classes of domains, in any number of variables. For (historical) comments we refer to [19].

Particularly relevant for potential and function theory are the solutions of the Fredholm eigenvalue problem:

$$K_\Omega u = 0.$$

They correspond to non-trivial solutions of the following *matching problem*: find analytic functions, continuous up to the boundary $f(z),\ z \in \Omega$, $g(z),\ z \in \hat{\mathbf{C}} \setminus \overline{\Omega},\ g(\infty) = 0$, such that:

$$f(\zeta) = \overline{g(\zeta)},\ \zeta \in \partial\Omega.$$

Non-trivial solutions exist on the *lemniscates* $\Omega = \{z \in \mathbf{C};\ |r(z)| < 1\}$, where r is a rational function satisfying $r(\infty) = \infty$. Indeed, it is clear that $f = r$ and $g = 1/r$ solve the above matching problem. For more details see [9].

The following result is reproduced (with its entire proof) from [28].

Theorem 4.1. *Let Ω be a connected generalized lemniscate of degree $d \geq 2$. Then* $\ker K_\Omega = 0$.

Proof. The proof is an adaptation of an argument, based on analytic continuation, from [9].

Write

$$\Omega = \{z \in \mathbf{C};\ \|(A - z)^{-1}\xi\| > 1\},$$

as in the preceding sections, where A is a $d \times d$ matrix with cyclic vector ξ.

We denote by $S(z)$ the d-valued Schwarz reflection, defined by the equation:

$$\langle (A - z)^{-1}x, (A - \overline{S(z)})^{-1}x \rangle = 1,\ z \in \mathbf{C}. \tag{4.1}$$

Let $a \in \partial\Omega$ be a non-ramification point for S. Since $\|(A - a)^{-1}x\| = 1$, then all local branches S_j of S satisfy

$$\|(A - \overline{S_j(a)})^{-1}a\| \geq 1, \quad 1 \leq j \leq d.$$

Denote $S_1(a) = \overline{a}$, so that $\|(A - \overline{S_1(a)})^{-1}x\| = 1$. Since every other branch has different values $S_j(a) \neq S_1(a)$ we infer that:

$$\|(A - \overline{S_j(a)})^{-1}x\| > 1, \quad 2 \leq j \leq d.$$

Therefore $\overline{S_j(a)} \in \Omega,\ 2 \leq j \leq d$, just as we proved in Proposition 3.4.

Similarly, for every exterior point b of $\overline{\Omega}$ we find that all branches satisfy $\overline{S_j(b)} \in \Omega,\ 1 \leq j \leq d$.

If the function $S(z)$ has no ramification points on $\tilde{\mathbf{C}} \setminus \overline{\Omega}$, then Ω is the complement of a quadrature domain (in the terminology of Aharonov and Shapiro) and the statement is assured by Theorem 3.19 of [9].

On the contrary, if the algebraic function $S(z)$ has ramification points in $\tilde{\mathbf{C}} \setminus \overline{\Omega}$, then, by repeating the main idea in the proof of the same Theorem 3.19 of [9], there exists a Jordan arc α starting at and returning to a point $a \in \partial\Omega$, such that $\alpha \setminus \{a\} \subset \tilde{\mathbf{C}} \setminus \overline{\Omega}$, having the property that the analytic continuation of $S_1(z)$ along this arc returns to a at another branch, say $S_2(z)$, with $\overline{S_2(a)}\Omega$.

Assume by contradiction that the matching problem on $\partial\Omega$ has a non-trivial solution: $f \in A(\Omega),\ g \in (\tilde{\mathbf{C}} \setminus \overline{\Omega}),\ g(\infty) = 0,\ f(\zeta) = \overline{g(\zeta)},\ \zeta \in \partial\Omega$. Hence

$$f(\overline{S_1(\zeta)}) = \overline{g(\zeta)},\ \zeta \in \partial\Omega.$$

Let us now describe with the point ζ the curve α, and after returning to the point a, describe $\partial\Omega$ once. By analytic continuation along this path, the matching condition continues to hold, and becomes:

$$f(\overline{S_2(\zeta)}) = \overline{g(\zeta)},\ \zeta \in \partial\Omega.$$

But now $\overline{S_2(\zeta)},\ \zeta \in \partial\Omega$ remains "trapped" into a compact subset $M \subset \Omega$. Thus, putting together the latter identities, we obtain:

$$\max_{\Omega} |f| = \max_{\Omega} |g| = \max_{M} |f|.$$

By the maximum principle, the function f should be a constant. Then g is a constant, too. But $g(\infty) = 0$, that is f and g are identically zero, a contradiction. □

Apparently there are no other examples of domains Ω with non-zero elements in the kernel of K_Ω other than the level sets of moduli of rational functions.

5. Root separation

The aim of this section is to show that the classical method of separating roots due to Hermite and later refined by Routh, Hurwitz, Schur, Cohn, Liénard and Chipard, and many other authors, can be combined with the specific form of the equation of a generalized lemniscate Ω, to obtain matricial criteria for the root location of an arbitrary polynomial, with respect to the domain Ω. The technical tools we invoke are elementary: the expression of the defining equation of the domain will be combined with some simple Hilbert space remarks; then Hermite's separation method (with respect to the half-space), or Schur's criterion (with respect to the disk) will be used. Along the same lines, more powerful methods based on the modern theory of the Bezoutiant can be exploited, [32].

Let

$$\Omega_R = \{z \in \mathbf{C};\ Q(z, \overline{z}) = 1 - \|R(z)\|^2 < 0\},$$

be a generalized lemniscate associated to a rational function

$$R : \mathbf{C} \longrightarrow \mathbf{P}_d,\ R(z) = (A - z)^{-1}\xi,$$

as considered in Section 3.

To start we remark that a point α belongs to $\mathbf{C} \setminus \overline{\Omega_R}$ if and only if, by definition, $\|R(\alpha)\| < 1$. In its turn, the latter condition is equivalent to $|\langle R(\alpha), v\rangle| < 1$ for all unit vectors $v \in \mathbf{C}^d$, or at least for the vectors of the form $v = R(\beta)/\|R(\beta)\|$, where β is not a pole of at least one, or a common zero of all, entries of R. Note that in the last formula $R(\alpha)$ depends rationally on the root α. Schur's criterion of separation with respect to the unit disk can then be applied, see for instance [3]. As a matter of fact the proof below allows us to consider slightly more general rational functions.

Theorem 5.1. *Let $R : \mathbf{C} \longrightarrow \mathbf{P}_d$ be a rational function satisfying $\lim_{z\to\infty} R(z) = 0$, and let $\Pi \subset \mathbf{C}$ be the set of all poles and common zeroes of R.*

Then a monic polynomial f has all its roots $\alpha_1, \ldots, \alpha_n$ in the open set $\mathbf{C} \setminus \overline{\Omega}_R$ if and only if, for every $\beta \in \mathbf{C} \setminus \Pi$, the polynomial

$$F_\beta(X) = \prod_{j=1}^{n} \Big(X - \frac{1 - Q(\alpha_j, \overline{\beta})}{\sqrt{1 - Q(\beta, \overline{\beta})}}\Big) \tag{5.1}$$

has all its roots in the unit disk.

Proof. Let f be a polynomial with all roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ in the set $\mathbf{C} \setminus \overline{\Omega}$. Then $\|R(\alpha_i)\| < 1$ for all $i, 1 \le i \le n$. Consequently, if $\beta \in \mathbf{C} \setminus \Pi$ we obtain:

$$|\langle R(\alpha_i), R(\beta)\rangle| < \|R(\beta)\|,$$

which is exactly condition in the statement.

Conversely, if the condition holds for all $\beta \in \mathbf{C} \setminus \Pi$, then by reversing the preceding argument we find that $\|R(\alpha_i)\| < 1,\ 1 \le i \le n$. □

Note that the polynomial $F_\beta(X)$ is a symmetric function of the roots $\alpha_j, 1 \le j \le n$, hence its coefficients are rational functions of $c_1, \ldots, c_n$. Therefore Schur's criterion will involve only rational combinations of the coefficients $c_1, \ldots, c_n$.

Specifically, if $F(z)$ is a polynomial with complex coefficients of degree d, we define the associated polynomials:

$$\overline{F}(z) = \overline{F(\overline{z})}, \qquad F^*(z) = z^d \overline{F}(\frac{1}{z});$$

then the inertia of the bilinear form:

$$G_F(X, Y) = \frac{F^*(X)\overline{F}^*(Y) - F(X)\overline{F}(Y)}{1 - XY},$$

gives full information about the root location of F with respect to the unit disk. That is, if G_F has d_+ positive squares and d_- negative squares, then the polynomial F has exactly d_+ roots in the unit disk, d_- roots outside the closed disk, and $d - d_+ - d_-$ roots lie on the unit circle.

Variations of the above result are readily available: for instance one can replace the rational map $R(z)$ by a polynomial map, or instead of F_β one can consider the polynomial involving the squares of the roots of F_β, and so on.

If we want to have more information about the root location of the polynomial $f(z) = (z - \alpha_1) \dots (z - \alpha_n)$, then the scalar products $\langle R(\alpha_j), v \rangle$, with v a fixed unit vector, can be replaced by an expression such as $\langle R(\alpha_j), \overline{S(\alpha_j)} \rangle$, where $S(z)$ is a vector valued rational function, of norm less than one in a large disk, where the roots are first estimated to be. Then, by counting parameters, the degree of S can be chosen to be dependent on n, the degree of f.

In order to state such a result, we make the following notation: for $S : \mathbf{C} \longrightarrow \mathbf{C}^d$ a vector valued rational map, let

$$F_S(X) = \prod_{j=1}^{n} (X - \langle R(\alpha_j), S(\overline{\alpha_j}) \rangle). \tag{5.2}$$

Note that this polynomial in X depends rationally on the entries α_j and is symmetrical in them. We have then

Corollary 5.2. *In the conditions of Theorem 2.2, let $U = t\mathbf{D}$ be a disk centered at the origin, that contains all the roots of the polynomial $f(z)$.*

Let $S : \mathbf{C} \longrightarrow \mathbf{C}^d$ be a rational map of degree less than or equal to s on each entry, satisfying $\|S(z)\| \leq 1, z \in U$, where we assume : $(2s+1)^d > dn$.

Then, with the above notations, we have:

$$|V(f) \cap \Omega_R| = \max_S |V(F_S) \setminus \overline{\mathbf{D}}|,$$

and

$$|V(f) \setminus \overline{\Omega_R}| = \min_S |V(F_S) \cap \mathbf{D}|.$$

Proof. Let $d_+ = |V(f) \setminus \overline{\Omega_R}|$ and $d_- = |V(f) \cap \Omega_R|$.

Since $\|S(\overline{\alpha_j})\| \leq 1$ for all $j, 1 \leq j \leq n$, we have $\langle R(\alpha_j), S(\overline{\alpha_j})\rangle) \leq \|R(\alpha_j)\|$. Therefore, the polynomial F_S has at least d_+ zeroes in the unit disk and at most d_- zeroes outside its closure.

To see that these bounds are attained, we remark that, due to the degree assumption, the map $S(z)$ can be chosen to have prescribed values at every point $\overline{\alpha_i}$, $1 \leq j \leq n$. Thus we can choose the values $S(\overline{\alpha_j})$ so that $\langle R(\alpha_j), S(\overline{\alpha_j})\rangle) = \|R(\alpha_j)\|$. □

Going into another direction, it is easy to establish *sufficient* criteria for the roots of the polynomial f to be all contained in the exterior of $\overline{\Omega}$. Let us denote the defining rational map by $R(z) = (R_1(z), \ldots, R_d(z))$.

Corollary 5.3. *In the conditions of Theorem 5.1, let $a_i, 1 \leq i \leq d$, be positive numbers satisfying $a_1^2 + a_2^2 + \ldots + a_d^2 = 1$.*

Define the polynomials:

$$F_i(X) = \prod_{j=1}^{n} (X - \frac{R_i(\alpha_j)}{a_i}), \quad 1 \leq i \leq d. \tag{5.3}$$

If the roots of each $F_i, 1 \leq i \leq d$, are contained in the unit disk, then the roots of f are contained in $\mathbf{C} \setminus \overline{\Omega}$.

Proof. It is sufficient to remark that, under the assumption for the roots of F_j, for each fixed $j, 1 \leq j \leq n$, we have

$$\|R(\alpha_j)\|^2 \leq \sum_{i=1}^{d} \|R_i(\alpha_j)\|^2 < \sum_{i=1}^{d} a_i^2 = 1$$

. □

For more details and a couple of examples see [25].

6. The reproducing kernel

Let $R(z) = (A - z)^{-1}\xi$ be a vector valued rational function attached as before to a matrix A and its cyclic vector $\xi \in \mathbf{C}^d$. The complement of the associated generalized lemniscate is a subset of the Riemann sphere

$$G = \{z \in \hat{\mathbf{C}};\ \|R(z)\| < 1\}.$$

As we have seen in Section 3, the map $R : G \longrightarrow B_d$ is a smooth rational embedding of G into the unit ball B_d of $\mathbf{C^d}$.

There are several reasons, simplicity and rationality being one of them, to consider the positive definite kernel

$$K(z,w) = \frac{1}{1 - \langle R(z), R(w) \rangle}, \quad z, w \in G.$$

The theory of such kernels is well understood but we will not invoke deep results about them; see [1], [4] for further applications of these kernels. We confine ourselves to reproduce from [25] and [5] an identification, as topological vector spaces, between the Hardy space of G and the reproducing Hilbert space with kernel K. This identification holds under a generic smoothness assumption and has some interesting function theoretic consequences.

Lemma 6.1. *Assume the domain G is simply connected and with smooth boundary. Then there are positive constants C_1, C_2 such that*

$$C_1 \sum_{j,k=1}^{N} \frac{\lambda_j \overline{\lambda_k}}{1 - z_j \overline{z_k}} \leq \sum_{j,k=1}^{N} \frac{\lambda_j \overline{\lambda_k}}{1 - \langle R(z), R(w) \rangle} \leq C_2 \sum_{j,k=1}^{N} \frac{\lambda_j \overline{\lambda_k}}{1 - z_j \overline{z_k}},$$

for every $N \in \mathbf{N}$ and choice of points $z_j \in G, \lambda_j \in \mathbf{C},\ 1 \leq j \leq N$.

The proof, based on the (Fredholm) analysis of the singular integral operator

$$f \mapsto \int_{\partial G} \frac{f(w)dw}{1 - \langle R(z), R(w) \rangle},$$

is contained in [5].

By passing for instance to the unit disk via a conformal mapping one deduces the following result.

Theorem 6.2. *Let $G = \{z \in \hat{\mathbf{C}};\ \|R(z)\| < 1\}$ be a smooth, simply connected domain. The Hilbert space with reproducing kernel $K(z,w)$ coincides as a set, but not isometrically in general, with the Hardy space of G.*

This fact was used in [5] to prove that a bounded analytic function along an analytic arc, smoothly attached to the unit sphere B_d, admits a bounded extension to the Schur class of B_d. This is a slight improvement of the known results of extending bounded analytic functions, from an attached disk, to a bounded analytic function defined on the whole ball.

7. Examples

The transition from the defining equation of a generalized lemniscate to its linear data is not totally trivial. Few concrete examples of this sort are known in full detail. We list below a couple of such elementary computations (appearing in [16]).

7.1. Domains corresponding to a nilpotent matrix.

A simple and basic example of a generalized lemniscate, which in general is not a quadrature domain, can be obtained as follows. Let us consider the nilpotent matrix A and the cyclic vector ξ :

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

where a, b, c are complex numbers, $c \neq 0$. A simple computation shows that:

$$\|(A-z)^{-1}\xi\|^2 = |\frac{a}{z} + \frac{b}{z^2} + \frac{c}{z^3}|^2 + |\frac{b}{z} + \frac{c}{z^2}|^2 + |\frac{c}{z}|^2.$$

Therefore the equation of the associated domain is:

$$|z|^6 < |az^2 + bz + c|^2 + |bz^2 + cz|^2 + |cz^2|^2.$$

According to Proposition 3.4, the reflection in the boundary of this domain maps the exterior completely into its interior.

The rational embedding associated to this example is:

$$R(1:z) = (-z^3 : az^2 + bz + c : bz^2 + cz : cz^2).$$

Similarly one can compute without difficulty the corresponding objects associated to a nilpotent Jordan block and an arbitrary cyclic vector of it. For instance the nilpotent $n \times n$-Jordan block and the vector $\xi = (0, 0, \ldots, 0, -1)$ give precisely the Veronese embedding:

$$R(1:z) = (z^n : 1 : z : \ldots : z^{n-2} : z^{n-1}).$$

7.2. The Limaçon.

This class of curves exhaust all quadrature domains of order two, with a double point.

Let us consider the conformal mapping $z = w^2 + bw$, where $|w| < 1$ and $b \geq 2$. Then it is well known that z describes a quadrature domain Ω of order 2, whose boundary has the equation:

$$Q(z, \overline{z}) = |z|^4 - (2 + b^2)|z|^2 - b^2 z - b^2 \overline{z} + 1 - b^2 = 0.$$

The Schwarz function of Ω has a double pole at $z = 0$, whence the associated 2×2-matrix A is nilpotent. Moreover, we know that:

$$|z|^4 \|(A-z)^{-1}\xi\|^2 = |z|^2 \|(A+z)\xi\|^2 = Q(z, \overline{z}).$$

Therefore

$$\|(A+z)\xi\|^2 = (2+b^2)|z|^2 + b^2 z + b^2\overline{z} + b^2 - 1,$$

or equivalently: $\|\xi\|^2 = 2+b^2, \langle A\xi, \xi\rangle = b^2$ and $\|A\xi\|^2 = b^2 - 1$.

Consequently the linear data of the quadrature domain Ω are:

$$A = \begin{pmatrix} 0 & \frac{b^2-1}{(b^2-2)^{1/2}} \\ 0 & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} \frac{b^2}{(b^2-1)^{1/2}} \\ (\frac{b^2-2}{b^2-1})^{1/2} \end{pmatrix}.$$

This shows in particular that the pair (A, ξ) is subject to some other restrictions than $A = 0$ and ξ being a cyclic vector for A.

The associated rational embedding can easily be computed from the definition:

$$R(1:z) = (-z^2 : \frac{b^2}{(b^2-1)^{1/2}} z + \frac{b^2-1}{(b^2-1)^{1/2}} : (\frac{b^2-2}{b^2-1})^{1/2} z).$$

7.3. Quadrature domains with two distinct nodes.

a). Suppose that Ω is a quadrature domain with the quadrature distribution:

$$u(f) = af(0) + bf(1),$$

where we choose the constants a, b to be positive numbers. Then $P(z) = z(z-1)$ and

$$z(z-1)(A-z)^{-1}\xi = -A\xi + \xi - z\xi.$$

Therefore the equation of the boundary of Ω is:

$$Q(z, \overline{z}) = |z(z-1)|^2 - \|A\xi - \xi + z\xi\|^2.$$

Accordingly we obtain:

$$\|\xi\|^2 = \frac{a+b}{\pi}, \qquad \langle A\xi, \xi\rangle = \frac{b}{\pi}.$$

Let us denote $\|A\xi\|^2 = c$. Then the defining polynomial becomes:

$$Q(z, \overline{z}) = |z(z-1)|^2 - \pi^{-1}(a|z-1|^2 + b(|z|^2 - 1)) - c.$$

The constant c actually depends on a, b, via, for instance, the relation $Area(\Omega) = a + b$, or, whenever $a = b$, the fact that $Q(1/2, 1/2) = 0$. The latter are called special points of a quadrature domain and were studied in [14].

We can choose an orthonormal basis with respect to which we have:

$$A = \begin{pmatrix} 0 & \alpha \\ 0 & 1 \end{pmatrix}, \qquad \xi = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}.$$

The matricial elements α, β, γ are then subject to the relations:

$$|\beta|^2 + |\gamma|^2 = \pi^{-1}(a+b), \quad \overline{\alpha}\beta\overline{\gamma} + |\gamma|^2 = \pi^{-1}b, \quad |\alpha|^2|\gamma|^2 + |\gamma|^2 = c.$$

An inspection of the arguments shows that the above system of equations has real solutions α, β, γ given by the formulas:

$$\alpha^2 = \frac{(\pi c - b)^2}{\pi(a+b)c - b^2},$$

$$\beta^2 = \frac{a^{-2}}{\pi(a-b) + \pi^2 c},$$

$$\gamma^2 = \frac{\pi(a+b)c - b^2}{\pi(a-b) + \pi^2 c}.$$

Let us remark that, if $a = b > \pi/4$, the constant c is effectively computable, as mentioned earlier, and becomes:

$$c = \frac{1}{16} + \frac{a}{2\pi}.$$

This again illustrates the special nature of the pair (A, ξ). A simple computation shows that the corresponding canonical embedding of the domain Ω is:

$$R(1:z) = (z(z-1) : \beta(1-z) - \alpha\gamma : \gamma z).$$

We remark that in both of the above examples, the matrix A and the vector ξ are uniquely determined, as soon as we require that A is upper triangular.

b). In complete analogy, we can treat the case of two nodes with equal weights as follows.

Assume that the nodes are fixed at ± 1. Hence $P(z) = z^2 - 1$. The defining equation of the quadrature domain Ω of order two with these nodes is:

$$Q(z, \overline{z}) = (|z+1|^2 - r^2)(|z-1|^2 - r^2) - c,$$

where r is a positive constant and $c \geq 0$ is chosen so that either Ω is a union of two disjoint open disks (in which case $c = 0$), or $Q(0,0) = 0$, as a special point property [14]. A short computation yields:

$$Q(z, \overline{z}) = z^2\overline{z}^2 - 2rz\overline{z} - z^2 - \overline{z}^2 + \alpha(r),$$

where

$$\alpha(r) = \begin{cases} (1-r^2)^2, & r < 1 \\ 0, & r \geq 1 \end{cases}$$

Exactly as in the preceding two situations, the identification

$$|P(z)|^2(1 - \|(A-z)^{-1}\xi\|^2) = Q(z, \overline{z}) \tag{7.1}$$

leads to (for example) the following simple linear data:

$$\xi = \begin{pmatrix} \sqrt{2}r \\ 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & \frac{\sqrt{2}r}{\sqrt{1-\alpha(r)}} \\ \frac{\sqrt{1-\alpha(r)}}{\sqrt{2}r} & 0 \end{pmatrix}.$$

7.4. Domains with rotational symmetry

One of the best understood classes of quadrature domains, and by extension, generalized lemniscates, is that of domains Ω invariant under a finite group of rotations

$$\Omega = \epsilon\Omega,$$

where ϵ is a primitive root of order n of unity.

Assume that the equation of Ω has the form:

$$Q(z,\overline{z}) = |P(z)|^2 - \sum_{k=0}^{d-1} |Q_k(z)|^2,$$

with $\mathrm{degP} = d$, $\mathrm{degQ_k} = k$. In view of the uniqueness of the above decomposition of Q, see Theorem 2.1, we infer:

$$|P(\epsilon z)| = |P(z)|$$

and

$$|Q_k(\epsilon z)| = |Q_k(z)|, \; 0 \le k \le d-1.$$

Assuming for instance that $n = d$, that is the order of the domain matches the symmetry order, we obtain:

$$Q(z,\overline{z}) = |z^d - a^d|^2 - \sum_{k=0}^{d-1} c_k |z^k|^2,$$

where $a \in \mathbf{C}$, $c_k > 0$, $0 \le k \le d-1$.

Among these domains only those for which $a = 0$ are quadrature domains. Computations along these lines around the uniqueness question, whether the quadrature data determine Ω, were carried out in [13], [14] and more recently in [7].

8. Disjoint unions of disks

In view of the preceding discussion, the constructive passage from an algebraic equation of a generalized lemniscate to its linear data is interesting and possibly useful for applications. We treat below a simple situation where the linear data can be recurrently computed.

A generic class of quadrature domains with positive weights in the associated quadrature formula is offered by the disjoint unions of disks. On fluid mechanical grounds any quadrature domain with positive weights has its roots in a disjoint union of disks. Although these sets are not connected, their equation is obviously within reach. We show below, following [17], how to compute inductively the associated linear data.

Lemma 8.1. *Let $D_i = D(a_i, r_i)$, $1 \le i \le n$, be disjoint disks and let*

$$Q(z, \overline{w}) = \prod_{i=1}^{n} [(z - a_i)(\overline{w} - \overline{a_i}) - r_i^2],$$

be the polarized equation defining their union. Then the matrix $(-Q(a_i, \overline{a_j}))_{i,j=1}^n$ is positive definite.

Proof. Let $\Omega = \cup_{i=1}^n D(a_i, r_i)$. Since the union is disjoint, Ω is a quadrature domain with nodes at $a_1, a_2, \ldots, a_n$. Let $P(z)$ be the monic polynomial vanishing at these points. According to Theorem 3.6, one can write:

$$Q(z, \overline{w}) = P(z)\overline{P(w)} - \sum_{k=0}^{d-1} |Q_k(z)|^2,$$

with polynomials Q_k of exact degree k, respectively.

Hence

$$Q(a_i, \overline{a_j}) = -\sum_{k=0}^{d-1} Q_k(a_i)\overline{Q_k(a_j)},$$

and this shows that the matrix in the statement is positive definite. □

A more detailed analysis shows that the matrix $(-Q(a_i, \overline{a_j}))_{i,j=1}^n$ is actually strictly positive definite, see [17].

We denote the same disjoint union of disks $\Omega_n = \cup_{i=1}^n D(a_i, r_i)$, and we consider the addition of an external disjoint disk; let $\Omega_{n+1} = \cup_{i=1}^{n+1} D(a_i, r_i)$ be the enlarged set. At each stage we have a finite dimensional Hilbert space K, a cyclic vector $\xi \in K$ and an operator $A \in L(K)$ which provide the linear data of these sets. We write accordingly, the equation of a disjoint union of k disks as:

$$Q_k(z, \overline{w}) = P_k(z)\overline{P_k(w)}[1 - \langle (A_k - z)^{-1}\xi_k, (A_k - w)^{-1}\xi_k \rangle],$$

where $A_k \in L(K_k)$ has cyclic vector ξ_k, $\dim K_k = k$, $k = n, n+1$, and the polynomial P_k has degree k and annihilates A_k.

Our aim is to understand the structure of the matrix A_{n+1} and its cyclic vector ξ_{n+1} as functions of the previous data (A_n, ξ_n) and the new disk $D(a_{n+1}, r_{n+1})$. Henceforth we assume that the closed disks $\overline{D(a_i, r_i)}$

are still disjoint. In order to simplify notation we suppress for a while the index $n+1$, e.g. $a = a_{n+1}$, $r = r_{n+1}$ etc. The following computations are based on standard realization techniques in linear systems theory.

Due to the multiplicativity of the defining equation for disjoint domains we find:

$$[1 - \langle (A_n - z)^{-1}\xi_n, (A_n - w)^{-1}\xi_n\rangle][1 - \frac{r^2}{(z-a)(\overline{w}-\overline{a})}] =$$
$$1 - \langle (A - z)^{-1}\xi, (A-w)^{-1}\xi\rangle.$$

Equivalently,

$$\langle (A_n - z)^{-1}\xi_n, (A_n - w)^{-1}\xi_n\rangle + \frac{r^2}{(z-a)(\overline{w}-\overline{a})} =$$
$$\langle \frac{r}{z-a}(A_n - z)^{-1}\xi_n, \frac{r}{w-a}(A_n - w)^{-1}\xi_n\rangle + \langle (A-z)^{-1}\xi, (A-w)^{-1}\xi\rangle.$$

Thus, for each z avoiding the poles, the norm of the vector

$$f(z) = \begin{pmatrix} (A_n - z)^{-1}\xi_n \\ \frac{r}{z-a} \end{pmatrix} \in K_n \oplus \mathbf{C}$$

equals that of the vector

$$g(z) = \begin{pmatrix} \frac{r}{z-a}(A_n - z)^{-1}\xi_n \\ (A-z)^{-1}\xi \end{pmatrix} \in K_n \oplus K.$$

And moreover, the same is true for any linear combination

$$\|\lambda_1 f(z_1) + \ldots + \lambda_r f(z_r)\| = \|\lambda_1 g(z_1) + \ldots + \lambda_r g(z_r)\|.$$

Because the span of $f(z)$, $z \in \mathbf{C}$, is the whole space $K_n \oplus \mathbf{C}$, there exists a unique isometric linear operator $V : K_n \oplus \mathbf{C} \longrightarrow K_n \oplus K$ mapping $f(z)$ to $g(z)$. We write, corresponding to the two direct sum decompositions

$$V = \begin{pmatrix} B & \beta \\ C & \gamma \end{pmatrix},$$

where $B : K_n \longrightarrow K_n$, $\beta \in K_n$, $C : K_n \longrightarrow K_{n+1}$, $\gamma \in K$. Since $Vf(z) = g(z)$ for all z, we find by coefficient identification:

$$B = r(A_n - a)^{-1}, \quad \beta = (A_n - a)^{-1}\xi_n.$$

The isometry condition $V^*V = I$ written at the level of the above 2×2 matrix yields the identities:

$$\text{(8.1)} \qquad \begin{cases} r^2(A_n^* - \overline{a})^{-1}(A_n - a)^{-1} + C^*C = I, \\ r(A_n^* - \overline{a})^{-1}(A_n - a)^{-1}\xi_n + C^*\gamma = 0, \\ \|(A_n - a)^{-1}\xi_n\|^2 + \|\gamma\|^2 = 1. \end{cases}$$

In particular we deduce that $(A_n^* - \overline{a})^{-1}(A_n - a)^{-1} \leq r^{-2}$ and since this operator inequality is valid for every radius which makes the disks disjoint, we can enlarge slightly r and still have the same inequality. Thus, the defect operator

$$\Delta = [I - r^2(A_n^* - \overline{a})^{-1}(A_n - a)^{-1}]^{1/2} : K_n \longrightarrow K_n \tag{8.2}$$

is strictly positive.

The identity $C^*C = \Delta^2$ shows that the polar decomposition of the matrix $C = U\Delta$ defines without ambiguity an isometric operator $U : K_n \longrightarrow K$. Since $\dim K = \dim K_n + 1$ we will identify $K = K_n \oplus \mathbf{C}$, so that the map U becomes the natural embedding of K_n into the first factor. Thus the second line of the isometry V becomes

$$(C \;\; \gamma) = \begin{pmatrix} \Delta & d \\ 0 & \delta \end{pmatrix} : K_n \oplus \mathbf{C} \longrightarrow K_n \oplus \mathbf{C} = K,$$

where $d \in K_n$, $\delta \in \mathbf{C}$. We still have the freedom of a rotation of the last factor, and can assume $\delta \geq 0$. One more time, equations (8.1) imply

$$\begin{cases} d = \frac{1}{r}(\Delta\xi_n - \Delta^{-1}\xi_n), \\ \delta = [1 - \|(A_n - a)^{-1}\xi_n\|^2 - \|d\|^2]^{1/2}. \end{cases} \tag{8.3}$$

From relation $Vf(z) = g(z)$ we deduce:

$$\begin{pmatrix} \Delta & d \\ 0 & \delta \end{pmatrix} \begin{pmatrix} (A_n - z)^{-1}\xi_n \\ \frac{r}{z-a} \end{pmatrix} = (A - z)^{-1}\xi.$$

This shows that $\delta > 0$ because the operator A has the value a in its spectrum.

At this point straightforward matrix computations lead to the following exact description of the couple $(A, \xi) = (A_{n+1}, \xi_{n+1})$ (by restoring the indices):

(8.4)

$$A_{n+1} = \begin{pmatrix} \Delta A_n \Delta^{-1} & -\delta^{-1}\Delta(A_n - a_{n+1})\Delta^{-1}d \\ 0 & a_{n+1} \end{pmatrix}, \quad \xi = \begin{pmatrix} \Delta^{-1}\xi_n \\ -\delta r_{n+1} \end{pmatrix}.$$

It is sufficient to verify these formulas, that is:

$$\begin{pmatrix} \Delta(A_n - z)\Delta^{-1} & -\delta^{-1}\Delta(A_n - a)\Delta^{-1}d \\ 0 & a - z \end{pmatrix} \begin{pmatrix} \Delta & d \\ 0 & \delta \end{pmatrix} \begin{pmatrix} (A_n - z)^{-1}\xi_n \\ \frac{r}{z-a} \end{pmatrix} =$$

$$\begin{pmatrix} \Delta^{-1}\xi_n \\ -\delta r \end{pmatrix}.$$

And this is done by direct multiplication:

$$\Delta\xi_n + \Delta(A_n - z)\Delta^{-1}\frac{rd}{z - a} - \Delta(A_n - a)\Delta^{-1}\frac{rd}{z - a} = \Delta^{-1}\xi_n,$$

which is equivalent to the known relation $dr = \Delta\xi_n - \Delta^{-1}\xi_n$.

Summing up, we can formulate the transition laws of the linear data of a disjoint union of disks.

Proposition 8.2. *Let* $\overline{D(a_i, r_i)}, 1 \leq i \leq n+1$, *be a disjoint family of closed disks, and let* $\Omega_k = \cup_{i=1}^k D(a_i, r_i),\ 1 \leq k \leq n+1$.

The linear data (A_k, ξ_k) *of the quadrature domain* Ω_k *can be inductively obtained by the formula* (8.4), *with the aid of the definitions* (8.2), (8.3).

Remark that letting $r = r_{n+1} \to 0$ we obtain $\Delta \to I$ and $d \to 0$, which is consistent with the fact that $\Omega_{n+1} \to \Omega$, in measure, in case such a limit domain Ω is given.

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Mathematics Department, University of California, Santa Barbara, CA 93106
E-mail address: mputinar@math.ucsb.edu