Semi-local micro-differential theory and computations of moments of semi-algebraic domains

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Abstract

In the extremal n-variable L-moment problem, the solution (=the characteristic function of $\{p < 0\}$,) p a polynomial, is determined by finitely many moments, in a set A.

In [23], for quadrature domains ($n = 2$), the reduction of all moments to the moments in A is implicit in the reconstruction of p using hyponormal operators.

For $n \geq 2$, assuming that the complexified of p has only isolated critical points for its singularities, we show that the local algorithm [22] which reduces to the moments in a base extends to the global case, with equality modulo analytic functions. The proofs depend on results from complex singularity theory and micro-differential systems ([15], [20], [21]).

0. Introduction

0.1. The (n-dimensional, $n \geq 1$) moment problem (in functional analysis) asks for a characterization of those numerical sequences $(a_\alpha)_{\alpha \in \mathbf{N}^n}$ that can be realized as the sequence of *moments* of a positive (Borel) measure μ supported on a subset of $\mathbf{R}^n$, i.e. s.t.

$$a_\alpha = a_\alpha(\mu) := \int x^\alpha d\mu(x),\ \alpha \in \mathbf{N}^n.$$

An obvious necessary condition is the non-negativity of the Hankel matrix $(a_{\alpha+\beta})_{\alpha,\beta \in \mathbf{N}^n}$. Versions of this condition are also shown to be sufficient under various conditions on the support of μ (cf. [7], [13], [23] and the references there).

A related problem, the extremal $L-$ moment problem on a fixed compact set in $\mathbf{R}^n$ (with L a bound on the norm of the measures sought, and with extremality in the set of all moments of such measures), is shown to have a (unique) solution (assuming the Hankel matrix ≥ 0 condition) of the form $\mu = \chi_{\{p<0\}}\, dx$ where $p \in \mathbf{R}[x_1, ..., x_n]$ and dx is the volume form $dx := dx_1 \wedge ... \wedge dx_n$ (χ denotes the characteristic function of a set).

In the "dynamic" setting, let us consider a family of relatively compact domains $(\Omega_t)_t$, $\Omega_t := \{p < t\}$, t a parameter, associated with a polynomial p, together with their moments $a_\alpha(t)$ with respect to the measure $\chi_{\Omega_t} dx$.

For $n \geq 2$, by the co-area theorem, we have, for t a regular value of p,

$$a'_\alpha(t) = \int_{\partial\Omega_t} x^\alpha dx/dp,$$

where $\omega := x^\alpha dx/dp$ is a C^∞ $(n-1)$-form s.t. $dp \wedge \omega = dx$, with uniquely defined restriction to $p^{-1}(t)$.

In this paper we obtain information on real integrals of the form

$$I_f(t) := \int_{\{p=t\}} f(x) dx/dp, \tag{1}$$

(and therefore on $a'_\alpha(t) = a'_{x^\alpha}(t)$), by considering instead complex integrals (in a complex variable t, associated with a complex polynomial -replacing p with its complexified $p_{\mathbf{C}}$ if p is real), of the form

$$I_f^\gamma(t) := \int_{\gamma(t)} f(x) dx/dp. \tag{2}$$

Here the variable $x = (x_1, ..., x_n)$ is extended to complex values; assuming $(*1)$, section 1.2, $[\gamma(t)] \in H^c_{n-1}(p^{-1}(t))$ is parallel transported (see section 1 for details).

In the local case, such integrals are studied extensively in singularity theory (cf. [1], [15],[19] etc.). For $n = 1$, the question of reduction of integrals to a basis is classical (cf. [8], [19]).

0.2. In the case of an isolated critical point for the complexified $p_{\mathbf{C}}$ of p, by adapting an algorithm (cf. [15]) that expresses n-forms in a certain free $\mathbf{C}[[\tau^{-1}]]$-module (τ a variable) in terms of a basis, the following local result was proved in [22].

Theorem 0.2.1. *Let $p = p_{\mathbf{C}} \in \mathbf{C}[x]$, $x = (x_1, ..., x_n)$, be s.t. 0 is a critical point for $p_{\mathbf{C}}$ where $p(0) = 0$. Let $(x^\alpha)_{\alpha\in I}$ be a monomial basis for $\mathbf{C}\{x_1, ..., x_n\}/I_{\nabla p}$, where $I_{\nabla p}$ = the ideal $(\partial_1 p, ..., \partial_n p)$. Let $f \in \mathbf{C}[x]$. Then*

a) for every $[\gamma(t)] \in H^c_{n-1}(p^{-1}(t))$ (parallel transported in the homological Milnor fibration),

$$I_f^\gamma = \sum_{\alpha\in I, 0\leq l\leq k} c^l_\alpha \partial_t^{-l} I_{x^\alpha}, \tag{3}$$

on a (punctured) neighborhood of $0 \in \mathbf{C}$. *Here* $k = k(f)$ *is finite and the coefficients* c_α^l *are computed by the algorithm of Lemma 0.2.2 below.*

b) if $p \in \mathbf{R}[x]$ *and if* $p \geq 0$ *or* 1 *is not an eigenvalue of the monodromy of* $p_{\mathbf{C}}$, *we have, asymptotically as* $t \to 0$ ($t > 0$, *resp.* $t < 0$)

$$a'_f(t) \sim \sum_{\alpha \in I, 0 \leq l \leq k} c_\alpha^l (\partial_t)^{-l} a'_{x^\alpha}(t), \tag{4}$$

mod a power series in t.

We shall denote the expression on the RH of (3) by

$$E_f^\gamma := \sum_{\alpha \in I, 0 \leq l \leq k} c_\alpha^l \partial_t^{-l} I_{x^\alpha}.$$

The notation ∂_t^{-1} will be used below mod $\mathcal{O}$ (cf. section 2). In the above Thm., it is an abbreviation for the algorithm proceeding backwards, from k to 0, with differentiation of the LH term instead of integration of the RH term (this gives a "continuous fraction" in ∂_t^{-1}).

Lemma 0.2.2. *The coefficients in the expression for* E_f *are computed starting with* $f_0 = f$ *by polynomial division and differentiation: for* $l \geq 0$,

$$f_l = \sum_{\alpha \in I} c_\alpha^l x^\alpha + \langle g^l, \nabla p \rangle, \quad f_{l+1} := \text{ div } g^l, \tag{5}$$

(here g^l *is a vectorial polynomial and* $\langle\ ,\ \rangle$ *denotes scalar product); the algorithm is finite since multideg* $(f_{l+1}) <$ *multideg* f_l.

We shall denote $k := \min \{\ l \mid f_{l+1} = 0\}$. The coefficients are unique (Lemma 2 of [PuG], argument using p, f real-valued). If $f \in I_{\nabla p}$ and γ is "generic" (i.e. its orbit under the monodromy group $\mathbf{C}$-generates $H_{n-1}(X_t)$, $X_t =$ the Milnor fiber) then $I_f^\gamma = 0$ if and only if $f(x)dx/dp$ is an exact form. (This characterizes most complex relations among derivatives of moments.) It is also useful to note for what follows that any writing as in Lemma 0.2.2 for f gives an algorithm as in Thm. 0.2.1 (possibly infinite though).

0.3. In the local case (at 0) one proves, more precisely (19]; cf section 2 for definitions) that the $\mathcal{O}_t$-submodule $\mathcal{K}_p^{(0)}$ of the Gauss-Manin micro-differential module $\mathcal{K}_p$ is free of rank μ over the ring $\mathbf{C}\{\{\partial_t^{-1}\}\}$ of micro-differential operators of order 0 with constant coefficients.

This implies that an $(n-1)$-form $\omega = f(x)dx/dp$ as above has a unique decomposition

$$f dx/dp = \sum_{\alpha \in I} P_\alpha \cdot x^\alpha dx/dp,$$

where $P_\alpha \in \mathbf{C}\{\{\partial_t^{-1}\}\}$ and $|I|$ = the Milnor number = dim $\mathbf{C}\{\mathrm{x}\}/(\partial_1 p, ..., \partial_n p)$.

Since $P_\alpha = \sum_{0 \le l < \infty} c_\alpha^l (\partial_t)^{-l}$, (with a certain convergence condition on the series of coefficients), integrating, this gives the expression for I_f^γ of Theorem 0.2.1 (but infinite, since $\mathbf{C}[x]$ is replaced with the local ring $\mathbf{C}\{x\}$.)

The assertion on the freeness of $\mathcal{K}_p^{(0)}$ includes therefore the fact that the coefficients are unique, and also the fact that the only relations possible are given by exact forms.

In the τ-variable, multiplication by τ^{-1} in $G_1 := \Omega^n / dp \wedge d(\Omega^{n-2})$ corresponds [Ph0] with the ∂_t^{-1}-action of $\mathbf{C}\{\{\partial_t^{-1}\}\}$ on $\mathcal{K}_p^{(0)}$. An integration by parts argument ([M1]) shows that the algorithm of Thm. 0.2.1 is essentially equivalent with writing in terms of a basis in each of these spaces.

0.4. In the global case, if we assume that p satisfies $(*1)$ (section 1), we can localize using results of [20] and prove (in section 3) the following.

Theorem 0.4.1. *Let $p, f \in \mathbf{C}[x]$, $x = (x_1, ..., x_n)$, where p satisfies (*1), and let $(x^\alpha)_{\alpha \in I}$ be a monomial basis for $\mathbf{C}[x]/I_{\nabla p}$, where $I_{\nabla p}$ = the ideal $(\partial_1 p, ..., \partial_n p)$ in $\mathbf{C}[x]$.*

Let $[\gamma(t)] \in H_{n-1}^c(p^{-1}(t))$ be obtained by parallel transport in the cohomology bundle over $\mathbf{C}^* := \mathbf{C} \setminus \mathcal{R}_p$, where $\mathcal{R}_p = \{t_j | 1 \le j \le s\}$ = is the set of critical values of p.

Then, with coefficients c_α^l as in Lemma 0.2.2, we have

a) $I_f^\gamma = E_f^\gamma$ (mod an entire function), for $t \in \mathbf{C}' := \mathbf{C} \setminus \cup_{1 \le j \le s} C_j$, where $(C_j)_j$ are parallel cuts at $(t_j)_j$ (in a generic direction θ).

b) if the solutions of the differential system $\mathcal{K}_p$ are of moderate growth at ∞, then the equality at a) becomes an equality mod a polynomial;

c) if p, f are real-valued and $p \ge 0$, then the moment $a_f(t)$ of $\Omega_t := \{p < t\}$ satisfies

$$a'_f(t) = \sum_{\alpha \in I, 0 \le l \le k} c_\alpha^l \partial_t{}^{-l} a'_\alpha(t)$$

mod a real-analytic function on $\mathbf{R}$, for $t >> 0$ (resp. $t << 0$).

The set I has cardinality μ_T :=total Milnor number of p = dim $_{\mathbf{C}}$ $\mathbf{C}[x]/I_{\nabla p}$ (cf. e.g. [10]).

Uniqueness of the coefficients of the algorithm of Theorem 0.4.1 follows since, by Lemma 3.5, I_f^γ decomposes uniquely as a sum of local $I_f^{e_j}$ mod $\mathcal{O}(\mathbf{C})$. For γ generic and $f \in I_{\nabla p}$, the relations among integrals are characterized by: $I_f^\gamma = 0$ iff for $1 \le j \le s$, $f dx/dp$ is exact on a neighborhood of the critical point x_j.

0.5. As to proofs, first, from the local algorithm of Theorem 0.2.1 and a decomposition theorem for n-chains ([20]), the global algorithm via the (Borel)-Laplace transform (in τ) follows immediately (Prop. 3.1).

In order to invert this (Prop. 3.8) to an algorithm in t, injectivity of the semi-local Laplace transform ([21]) under (various) growth conditions ([17]) is needed; with moderate growth conditions, pnt. b) of thm. 0.4.1 follows.

A direct proof (Prop. 3.4) of pnt. a) follows from the local algorithm in t by the variation map ([20]) and a standard local cohomology argument (Lemmas 3.5, 3.6).

In Prop. 3.2 (using [20] -the variation map and (Borel)-Laplace transforms), we prove a version of the local algorithm, for co-vanishing instead of vanishing cycles.

Finally, we apply the above to the real case, to obtain an asymptotics algorithm for oscillatory integrals with phase p (Proposition 3.10); if $p \geq 0$, we obtain as a consequence (Proposition 3.11) the real semi-local algorithm (by the co-area theorem, this proves pnt. c) of Thm. 0.4.1).

These technical results on the semi-local Gauss-Manin are in the spirit of [21], but are different from them in several ways (cf. Remark 3.7). The author could not find any written account of these nor any up-dates in connection with this particular direction. Quite remotely, for extremal moments, the question of reducing moments to a basis was asked in [23] and solved (implicitly, in the reconstruction algorithm for p) for $n = 2$ (in several cases, including that of quadrature domains), by using hyponormal operators.

0.6. The paper is organized as follows: section 1 consists of definitions and properties of the various integrals associated with the polynomial p. In section 2, the complex integrals are viewed as solutions of the Gauss-Manin (micro)-differential system.(These two sections are preliminary). In section 3 we prove several versions of the algorithm of theorem 0.4.1 in the global case. Section 4 consists of examples corroborating the algorithm with asymptotic computations ([4], [18]) and with computations for planar quadrature domains ([2], [12]).

1. Integrals (relative a polynomial p).

1.1. Moments and the Dirac distribution. If $p \geq -M$ (some constant M) is a polynomial in $\mathbf{R}[x] = \mathbf{R}[x_1, ..., x_n]$, then the moments $a_f(t)$ of $p < t$ are well-defined (for f polynomial) and $I_f(t) = a'_f(t)$ for t a regular value of p (cf. 0.1). (The form $f dx/dp$ in (1) denotes the restriction to $p^{-1}(t)$ of any $(n-1)$-form η s.t. $dp \wedge \eta = f dx$ - this is independent of the choice of η.)

Let us now consider the local case of domains in a neighborhood of a critical value (0, say) of p. The integral (1) is well-defined if $f \in C_0^\infty$ for p arbitrary in $\mathbf{R}[x]$.

The map from C_0^∞ to $\mathbf{R}$ defined by $f \to I_f$ is the "Dirac distribution along p". It has an asymptotic expansion at (the critical value) 0, with coefficients distributions in f, of the form:

$$I_f(t) \sim \sum_{\alpha_0 > -1, 0 \leq k \leq n-1} c_{\alpha_0,k}(f) \cdot t^{\alpha_0} (\log t)^k,$$

where the index α_0 ranges over a sequence of rational numbers (with a single denominator, depending on p only).

If 0 is an isolated critical value for p, then the asymptotic expansion as $t \to 0$ of the Dirac distribution applied at f depends only on the Taylor series of f at 0; if moreover 0 is an isolated critical value for $p_{\mathbf{C}}$, then the indices α_0, k depend on the monodromy of $p_{\mathbf{C}}$ (cf. [15]).

1.2. Homology decompositions. Let $p : \mathbf{C}^n \to \mathbf{C}$ be a polynomial. Then p has finitely many branching points (i.e. critical values and "second type" singularities (cf. [20] for the definition).

We shall assume that the following *condition (*1)* is satisfied:

i) p has no "second type" singularities ([20]); and
ii) for every critical value there is only one corresponding critical point.

We shall denote the (finite) set of critical points (resp. critical values) of p by

$$\mathcal{C}_p = \{x_j \mid 1 \leq j \leq s\}, \text{ resp.} \mathcal{R}_p = \{t_j \mid 1 \leq j \leq s\}. \tag{6}$$

By (*1), p defines a locally trivial fibration over $\mathbf{C}^* := \mathbf{C} \setminus \mathcal{R}_p$. The parallel transport of homology classes is well-defined in $\mathbf{C}^*$ and by [20] we can also localize homology classes, w.r.t. a fixed generic direction of angle θ, as follows.

With notation $S_c^+ = S_c^+(\theta) = \{t \in \mathbf{C} \mid \text{ Re } te^{-i\theta} \geq c\}$, (resp. S_c^- with $\geq$ changed to $\leq$), letting $\Phi = \Phi(\theta) := \{A \text{ closed } \subset \mathbf{C}^n \mid A \cap p^{-1}(S_c^-) \text{ compact},$ for some $c > 0\}$, the cohomology groups $H_n^\Phi(\mathbf{C}^n) := \lim_{\leftarrow} H_n(\mathbf{C}^n, p^{-1}(S_c^+))$ are defined ([20]) using semi-algebraic chains. For $c >> 0$ (sufficient c s.t. all critical values of p are in S_c^-), and if $t \in S_c^+$, the boundary map $\partial : H_n^\Phi(\mathbf{C}^n) \to H_{n-1}^c(p^{-1}(t))$ is an isomorphism.

We denote by H^c or H homology with compact supports; the coefficients are taken in $\mathbf{C}$.

By a deformation retract and excision argument, it follows that $H_n^\Phi(\mathbf{C}^n) \cong \oplus_{1 \leq j \leq s} H_n(X_j, X_j^+)$, where X_j is a Milnor ball at the critical point x_j and $X_j^+ := p^{-1}(S_j) \cap X_j$, with notation $S_j^+ = t_j + S_\delta^+$ (for a certain small $\delta > 0$.) To this isomorphism corresponds a decomposition

$$\Gamma = \sum_{1 \leq j \leq s} \Gamma_j, \tag{7}$$

for $\Gamma \in H_n^\Phi(\mathbf{C}^n)$.

Applying the boundary map (isomorphism), a similar decomposition holds for $(n-1)$-cycles. If $\gamma \in \mathrm{H}_{n-1}^c(p^{-1}(t))$, then

$$\gamma = \sum_{1 \leq j \leq s} e_j, \tag{8}$$

where e_j is a cycle vanishing in direction θ, i.e. an element in $H_{n-1}(X_{t_{0j}})$, for t_{0j} arbitrary $\in S_j^+$.

1.3. Complex integrals. Let p, f be complex polynomials (n variables), p satisfying $(*1)$. For $[\gamma(t)] \in H^c_{n-1}(p^{-1}(t))$ parallel transported in the cohomology bundle, the complex integrals (2) are well-defined and holomorphic multi-valued for $t \in \mathbf{C} \setminus \mathcal{R}_p$.

From the definitions with forms, if p is real the (complex) integral $I_f^\gamma(t)$ coincides with the (real) integral $\mathrm{I}_f(t)$ for t real. This is so for instance if $p \geq 0$, since in this case $\gamma(t) = p_{\mathbf{C}}^{-1}(t) \cap \mathbf{R}^n$ is compact.

For $\Gamma \in H_n^\Phi(\mathbf{C}^n)$, ($\Phi = \Phi(\theta)$ as in 1.2), the integrals with *phase* p and *amplitude* f (both complex polynomials) are well-defined [20] (i.e. convergent; besides obviously holomorphic in τ) by

$$I_f^\Gamma(\tau) := \int_\Gamma e^{-\tau p(z)} f(z) dz, \tag{9}$$

We consider next localized versions at a critical value (0, say) of the above integrals.

Let $X_t =$ the Milnor fibre $:= B \cap p^{-1}(t)$, t in a small disc D s.t. $0 \in D$. Let $H^F_{n-1}(X_t)$ denote cohomology with closed supports; the elements of $H^c_{n-1}(X_t)$ (resp. $H^F_{n-1}(X_t)$) are called vanishing (resp. co-vanishing) cycles.

By replacing, in the above integrals, $\gamma(t)$ with $e(t) \in H^c_{n-1}(X_t)$ (resp. $\epsilon(t) \in H^F_{n-1}(X_t)$), we obtain $I_f^e(t)$ (resp. $I_f^\epsilon(t)$).

These integrals are shown to be holomorphic multi-valued in $D^* = D \setminus 0$ (resp. holomorphic multi-valued in D^* mod holomorphic in D). We shall use the standard notations $\tilde{\mathcal{O}}(D)$ (resp. $\tilde{\mathcal{O}}(D)/\mathcal{O}(D)$) for the set(s) of such functions.

1.4. (Borel)-Laplace transforms. In the local case ($0 =$ critical point for p, $p(0) = 0$), the complex integrals defined above are related as follows [20].

Let $\epsilon = \epsilon_+ \in H^F_{n-1}(X_{t+})$, $t_+ > 0$ in neighborhood of 0, be a co-vanishing cycle (corresponding to direction $\theta = 0$). The cycle $e_+ =$var$(\epsilon) \in H^c_{n-1}(X_{t_+})$ is defined by $\mathrm{var}(\epsilon) = \epsilon' - \epsilon$, where ϵ' is obtained by parallel transporting ϵ along a circuit (counter-clockwise) around 0. This generates an isotopy of the Milnor ball X (s.t. $=$ identity on ∂X) and with it an n-dimensional chain $\Delta_+ \in H_n(X, X^+)$ s.t. $\partial_+ \Delta_+ = e_+$.

Relative to these paths of integration, it is shown (with definitions below) that

$$\int_{\Delta_+} e^{-\tau p} \omega = \mathcal{BL}(I_f^{e_+})(\tau) = \mathcal{L}(I_f^{\epsilon_+})(\tau), \tag{10}$$

for every holomorphic n-form $\omega = f(z)dz$, with equality of convergent integrals for τ in a sector in $\mathbf{C}$.

Here $\mathcal{BL}$ denotes the Borel-Laplace transform, well-defined by

$$\mathcal{BL}(\tilde{h})(\tau) := \int_0^\infty e^{-\tau t} \tilde{h} dt,$$

for $\tilde{h}$ holomorphic multi-valued on a sector with origin at 0.

The Laplace transform $\mathcal{L}$ is defined for $\tilde{h}$ as above by

$$\mathcal{L}(\tilde{h}) := \int_{\delta} e^{-\tau t}\tilde{h}dt + \int_{\sigma} e^{-\tau t}\mathrm{var}(\tilde{h}), \tag{11}$$

where σ (resp. δ) is a small circle, oriented clockwise, with base point t_0 near 0 (resp. a half-line in direction $e^{i\theta}$, with origin t_0), and $\mathrm{var}(\tilde{h})$ is the variation map on functions:

$$\mathrm{var}\ (\tilde{h})(t) = \tilde{h}(e^{-2\pi i} \cdot t) - \tilde{h}(t). \tag{12}$$

2. (Micro)-differential (semi)-local theory.

2.1. Solutions. Let $n \geq 1$, $x = (x_1, ..., x_n)$ be the variable in $\mathbf{C}^n$ and let $p : \mathbf{C}^n \to \mathbf{C}$ be a polynomial s.t. (*1). For the local study, assume that 0 is an isolated critical point of p of Milnor number μ and that $p(0) = 0$. Let X_t be the Milnor fibre at 0, with t a regular value (near 0) for p.

In the differential case, for $e(t) \in H_{n-1}(X_t)$ a vanishing cycle, the integral $I_f^e(t)$ (cf. 1.3) is a solution (applied at a differential form $f(x)dx/dp$) of the differential system $\mathcal{K}_p$ defined below (the solutions of which are obtained by integrating over slightly more general forms).

Let $\Omega^*_{x,t/t}$ $(* \in \mathbf{N})$, denote the complex of (germs of) relative differential forms (consisting of forms of the type $\sum_\alpha a_\alpha(x,t)dx^\alpha$ with holomorphic coefficients).

Let

$$\Omega^*_p := \Omega^*_{x,t/t}[\frac{1}{t-p}]/\Omega^*_{x,t/t}$$

be the complex of meromorphic differential forms with pole along the graph of p.

Let $\mathcal{D}_t$ (resp. $\mathcal{D}_{t,x}$) be the ring of (germs at 0) of differential operators in t (resp. (t,x)) with analytic coefficients.

The *differential Gauss-Manin module* is the $\mathcal{D}_t$-module defined by

$$\mathcal{K}_p := \Omega^n_p / d\Omega^{n-1}_p. \tag{13}$$

The $\mathcal{D}_t$-structure of $\mathcal{K}_p$ is induced on forms in Ω^n_p from the derivation ∂_t on integrals over the forms. (By the residue map, it coincides with the Gauss-Manin connexion).

The function

$$t \to I(\zeta)(t) := \int_{e(t)} \mathrm{Res}\ \zeta$$

has its values in $\tilde{\mathcal{O}}_{t,0}$(= the space of germs at 0 of multi-valued analytic functions) and the map $\int_{e(t)} : \zeta \to I(\zeta)$ is a *solution* of $\mathcal{K}_p$, i.e. an element of $\mathrm{Hom}_{\ \mathcal{D}_t}(\mathcal{K}_p, \tilde{\mathcal{O}}_{t,0})$.

The homomorphisms $\int_{e(t)}$ as $e(t)$ varies in $H_{n-1}(X_t)$ are all the solutions of $\mathcal{K}_p$ (and in a one-to-one correspondence with the vanishing cycles $e(t)$).

If ζ has a simple pole, writing $\zeta = \omega/2\pi i(p-t)$, with ω a holomorphic n-form, its residue is Res $\zeta = \omega/dp|_{p^{-1}(t)}$. Writing $\omega = f dx$, with f a holomorphic function, the integral $I_f^e(t)$ is therefore a solution (applied at ζ) of $\mathcal{K}_p$. The set of such integrals forms a *lattice* $\mathcal{K}_p^0$ in $\mathcal{K}_p$, i.e. an $\mathcal{O}_t$-module that generates $\mathcal{K}_p$ (freely, with rank μ) as a $\mathcal{D}_t$-module.

In the micro-differential case, if $\epsilon(t) \in H_{n-1}^F(X_t)$, the integral $I_f^\epsilon(t)$ is a solution (applied at a differential form) of the micro-differential system $\mathcal{K}_p$.

This system is an $\mathcal{E}_t$-module with structure extending its $\mathcal{D}_t$-module structure, where $\mathcal{E}_t$ - the ring of *micro-differential operators* - is a ring containing $\mathcal{D}_t$ in which ∂_t is invertible ($\mathcal{E}_t =$ the micro-localization of $\mathcal{D}_t$).

The submodule $\mathcal{K}_p^{(0)}$ is free of rank μ over the ring $\mathbf{C}\{\{\partial_t^{-1}\}\}$ of micro-differential operators of order 0 with constant coefficients.

2.2. Semi-local theory. The study of semi-local solutions is described in [21], in the case of (non-characteristic) systems of the form $\mathcal{E}_{t,x}/\mathcal{I}$, where $\mathcal{I}$ is an ideal (t, x are complex variables, $x = (x_1, ..., x_n)$).

In the case of a single variable t, since $\mathcal{E}_t$ is a p.i.d., it is possible to reduce to a system defined by a single differential polynomial of the form

$$P(t, \partial_t^{-1}) = t^m + \sum_{0 \le k \le \infty} c_k(t) \partial_t^{-k},$$

with a certain condition of simultaneous convergence for $c_k \in \mathcal{O}_{t,0}$. Here $m \in \mathbf{N}$ is the *multiplicity* of P.

The space of solutions relative a fixed disk D in $\mathbf{C}$ is defined (for U simply-connected open in D) by

$$\text{Sol}\ ^D(U) = \{u \in \mathcal{O}(U) \mid P_{t_0}(u) \in \mathcal{O}(\overline{D})\},$$

(where the action of ∂_t^{-1} in P_{t_0} is taken by integrating w.r.t. to t_0, a base point in D - here the convergence conditions ensure this is possible for some D; the space Sol is independent of the choice of t_0).

The space of "singularities" of solutions is $\text{sol}^D(U) := \text{Sol}^D(U)/\mathcal{O}(\overline{D})$. Both Sol and sol are regarded as sheaves and germs of solutions can be considered. In a semi-local situation one replaces t^m with a small perturbation of it, a polynomial of degree m having all its roots $(t_j)_{1 \le j \le m}$ in D. Fixing a generic direction and (parallel) cuts C_j at t_j in that direction, the solutions are uni-valued on

$$D' = D \setminus \cup_{1 \le j \le m} C_j. \tag{14}$$

An argument based on index for operators shows that local solutions at a point extend to D' and that $\dim_{\mathbf{C}} \text{sol}^D(D') =$ m.

From this, the following decomposition of the space of semi-local solutions (mod $\mathcal{O}$) is obtained:

$$\text{sol }^D(D') = \oplus_{1\leq j\leq m}\text{sol }^{D_j}(D'_j), \tag{15}$$

where D_j is a small disk at t_j and $D'_j = D_j \setminus C_j$.

Further, the Laplace transform on the space sol is injective and maps the summands of this decomposition to subspaces of $\mathcal{A}^{\leq r}/\mathcal{A}_{\leq -r}$ (growth $\leq Ce^{r|t|}$ resp. $\leq Ce^{-r'|t|}$, where $r = (1/\sqrt{2})\cdot$ radius of the disk and $r' < r$ is variable).

The (micro-differential) Gauss-Manin module $\mathcal{K}_p$ (discussed at 2.1) has a presentation as an $\mathcal{E}_t$-module of the form

$$\mathcal{K}_p \cong \mathcal{B}_p / \sum_{1\leq i\leq n} \partial_{x_i}\mathcal{B}_p,$$

where

$$\mathcal{B}_p := \mathcal{E}_{t,x}/\mathcal{E}_t\cdot(t-p, \partial_{x_i} - \partial_{x_i}p\cdot\partial_t).$$

The module $\mathcal{K}_p$ is not of the form $\mathcal{E}_{t,x}$ considered above and while in principle this case is sufficient for the study of general holonomic systems ([17]), it is possible to obtain (cf. section 3) a similar semi-local decomposition formula by other, simpler means.

3. Algorithm for moments - semilocal / global case

Let $p \in \mathbf{C}[x] = \mathbf{C}[x_1, ..., x_n]$, $n \geq 1$, satisfy $(*1)$, let (cf. 1.2) $\mathcal{C}_p$, $\mathcal{R}_p$ denote its set of critical points (resp. critical values) and let $\theta \in \mathbf{R}$ be the angle of a fixed generic direction in $\mathbf{C}$.

Proposition 3.1. *Let $p \in \mathbf{C}[x] = \mathbf{C}[x_1, ..., x_n]$, $n \geq 1$, satisfy (*1) and let $(x^\alpha)_{\alpha\in I}$ be a monomial basis for $\mathbf{C}[x]/I_{\nabla p}$. Let $\Gamma \in H_n^\Phi$, $\Phi = \Phi(\theta)$ (cf. 1.2).*

Then for every $f \in \mathbf{C}[x]$, there exists a finite k (depending on p and f) such that (cf. 1.3 for notations)

$$I_f^\Gamma(\tau) = \sum_{\alpha\in I, 0\leq l\leq k} c_\alpha^l \left(\frac{1}{i\tau}\right)^l I_\alpha^\Gamma(\tau) \tag{16}$$

with equality asymptotically in $\tau \to \infty$ (τ in a sector containing a half-line in direction $e^{i\theta}$). The coefficients $c_\alpha^l \in \mathbf{C}$ are computed by the algorithm of Lemma 0.2.2.

Proof. By [20] (cf. 1.2), Γ has a decomposition $\Gamma = \sum_{1\leq j\leq s}\Gamma_j$, where Γ_j is admissible, i.e. in $H_n(X^j, X^{+j})$ (where $+$ is w.r.t direction θ). By the local algorithm (Thm. 0.2.1) in τ for the integral over Γ_j (relative any writing as in Lemma 0.2.2, not necessarily w.r.t. a local basis), we have

$$I_f^{\Gamma_j}(\tau) = \sum_{\alpha \in I, 0 \le l \le k} c_\alpha^l (\frac{1}{i\tau})^l I_\alpha^{\Gamma_j}(\tau), \tag{17}$$

asymptotically as $\tau \to \infty$. Noting that the same form $(f dx - \sum_{\alpha \in I, 0 \le l \le k} c_\alpha^l x^\alpha dx)$ is integrated at all the critical points of p, and since $I_f^\Gamma = \sum_{1 \le j \le s} I_f^{\Gamma_j}$, summing over j in the above formula gives formula (16) for Γ. □

We have the following algorithm mod $\mathcal{O}$ in the local case, for integrals over *co-vanishing* cycles.

Proposition 3.2. *Let p, $f \in \mathbf{C}[x]$, and let 0 be a critical point for p. Let $\epsilon(t) \in H_{n-1}^F(p^{-1}(t) \cap X)$, ($X \to D =$ Milnor fibration at 0) be defined by parallel transportation. Then*

$$I_f^\epsilon(t) = E_f^\epsilon(t) := \sum_{\alpha \in I, 0 \le l \le k} c_\alpha^l (\partial_t)_{t_{0+}}^{-l} I_\alpha^\epsilon(t) \ mod\ \mathcal{O}(D), \tag{18}$$

(multi-valued) for t in $D^* = D \setminus \{0\}$ (and any choice of the base point t_{0+}); the coefficients $c_\alpha^l \in \mathbf{C}$ are computed by the algorithm of Lemma 0.2.2.

Proof. By the local algorithm (cf. 0.2.1) for $e :=$var(ϵ) (a vanishing cycle), we have $I_f^e(t) = E_f^e(t)$ for $t \in D^*$. Therefore also $\mathcal{BL}(I_f^e)(\tau) = \mathcal{BL}(E_f^e)(\tau)$ for τ in an infinite sector. By [20] (cf. 1.4), this is equivalent with $\mathcal{L}(I_f^\epsilon)(\tau) = \mathcal{L}(E_f^\epsilon)(\tau)$, from which it follows that $I_f^\epsilon(t) = E_f^\epsilon(t)$ mod $\mathcal{O}(D)$, since the local Laplace transform is an injection on O(S)/O(D)(cf. [17], [21]), where S is a sector in D. □

Notations 3.3. (In $\mathbf{C}$, w.r.t. $\mathcal{R}_p$ and a fixed θ.). For $R > 0$, let $S = S_R := \{|t| < R|\ \mathrm{Re}(te^{-i\theta}) < 0\} \cup \{t \mid\ \mathrm{Re}(te^{-i\theta}) \ge 0$ and $|\ \mathrm{Im}(te^{-i\theta})| < R\}$ (i.e. a half-disk $\cup$ a half-strip), let $C_j = \{\ \mathrm{Re}(t - t_j)e^{-i\theta} \ge 0\}$ be parallel cuts at t_j. Let S_j $(1 \le j \le s)$ be open "thickenings" of C_j containing t_j. Then $C_j \subset S_j \subset S$, for $1 \le j \le s$.

Let $S' := S \setminus \cup_{1 \le j \le s} C_j$ and $S'_j := S_j \setminus C_j$; $\mathbf{C}' := \mathbf{C} \setminus \cup_{1 \le j \le s} C_j$, $\mathbf{C}^* = \mathbf{C} \setminus \mathcal{R}_p$; similarly D', D^*, for D a disk.

Proposition 3.4. *Let p, $f \in \mathbf{C}[x]$, where p satisfies $(*1)$ and let $\gamma = [\gamma(t)] \in H_{n-1}^c(p^{-1}(t))$ be defined by parallel transportation, univalued for $t \in \mathbf{C}'$ (cf. 3.3, w.r.t. a direction θ). Then*

$$I_f^\gamma(t) = E_f^\gamma(t) := \sum_{\alpha \in I, 0 \le l \le k} c_\alpha^l (\partial_t)^{-l} I_\alpha^{\gamma(t)} \ mod\ \mathcal{O}(\mathbf{C}), \tag{19}$$

(uni-valued) for t in $\mathbf{C}'$; the coefficients $c_\alpha^l \in \mathbf{C}$ are computed by the algorithm of Lemma 0.2.2.

Proof. a) Since by $(*1)$, p defines a locally trivial fibration over $\mathbf{C}'$, $[\gamma(t)]$ can be transported by parallelism to $[\gamma(t_0)]$, $t_0 \in S_c^+(\theta)$, $c >> 0$.

By [PhI], $[\gamma(t_0)] = \sum_{1\leq j\leq s}[e_j]$, $e_j \in H_{n-1}^c(X_{t_{0j}})$, vanishing cycles at t_j (t_{0j} in X_j^+, $+$ being w.r.t. direction θ).

b) By the local algorithm (Thm. 0.2.1) w.r.t. direction θ, $I_f^{e_j}(t) = E_f^{e_j}(t)$, for $1 \leq j \leq s$, t in X_j^+.

c) For $S = S_R$ (notations as in 3.3) there is a (natural) decomposition (cf. Lemma 3.5 below)

$$\mathcal{O}(S')/\mathcal{O}(S) = \oplus_{1\leq j\leq s}\mathcal{O}(S_j')/\mathcal{O}(S_j).$$

For $t \in S'$, by a) and Lemma 3.6 below

$$I_f^\gamma(t) = \sum_{1\leq j\leq s} I_f^{e_j}(t) \bmod \mathcal{O}(S)$$

and similarly

$$E_f^\gamma(t) = \sum_{1\leq j\leq s} E_f^{e_j}(t) \bmod \mathcal{O}(S).$$

Using b), it follows that $\sum_{1\leq j\leq s} I_f^{e_j}(t) = \sum_{1\leq j\leq s} E_f^{e_j}(t) \bmod \mathcal{O}(S)$ for $t \in S'$. Therefore also, $I_f^\gamma(t) = E_f^\gamma(t) \bmod \mathcal{O}(S)$, for $t \in S'$. Letting $R \to \infty$, since solutions extend to $\mathbf{C}^*$, we have equality mod $\mathcal{O}(\mathbf{C})$. □

Using (8), from Lemmas 3.5 and 3.6 below it will follow that the integral $I_{f(z)}^\gamma(t)$ over a global cycle $[\gamma] \in H_{n-1}(p^{-1}(t))$, $t \in S_c^+(\theta)$, $c >> 0$, localizes, i.e.

$$I_f^\gamma(t) = \sum_{1\leq j\leq s} I_f^{e_j}(t) \bmod \mathcal{O}(\mathbf{C}).$$

We prove this by using standard arguments from local cohomology.

Lemma 3.5. *With notations as in 3.2, we have*

$$\mathcal{O}(S')/\mathcal{O}(S) \cong \oplus_{1\leq j\leq s}\mathcal{O}(S_j')/\mathcal{O}(S_j) \tag{20}$$

Proof. Since $Z := \cup_{1\leq j\leq s} C_j$ is closed in S, we have the following exact sequence in local cohomology with supports in Z ([9], cf. also [3]).

$$0 \to H_Z^0(S,\mathcal{O}) \to H^0(S,\mathcal{O}) \to H^0(S\setminus Z, \mathcal{O}|_{S\setminus Z}) \to H_Z^1(S,\mathcal{O}) \to H^1(S,\mathcal{O}) \to ...$$

The leftmost term =0, by the identity principle for (one variable) holomorphic maps; the rightmost term =0 since S (= an open domain in $\mathbf{C}$) is Stein. Hence the exact sequence

$$0 \to \mathcal{O}(S) \to \mathcal{O}(S') \to H^1_Z(S,\mathcal{O}) \to 0,$$

or equivalently

$$\mathcal{O}(S')/\mathcal{O}(S) \cong H^1_Z(S,\mathcal{O}).$$

Similarly,

$$\mathcal{O}(S'_j)/\mathcal{O}(S_j) \cong H^1_{C_j}(S_j,\mathcal{O}_{|S_j}).$$

On the other hand, by excision for local cohomology one has

$$H^1_Z(S,\mathcal{O}) \cong \oplus_{1\le j\le s} H^1_{C_j}(S,\mathcal{O})$$

and since C_j is closed in S_j and S_j is open in S, also

$$H^1_{C_j}(S,\mathcal{O}) \cong H^1_{C_j}(S_j,\mathcal{O}_{|S_j}).$$

From this the lemma follows immediately.□

Lemma 3.6. *Let p satisfy $(*1)$ and let S (cf. notations 3.3) contain $\mathcal{R}_f$. Let $[\gamma] = \sum_{1\le j\le s}[e_j]$ in $H^c_{n-1}(p^{-1}(t_o)) \cong \oplus_{1\le j\le s} H^c_{n-1}(X^+_{t_j+})$.*

Then $I^\gamma_f(t) = \sum_{1\le j\le s} I^{e_j}_f(t)$ mod $\mathcal{O}(S)$ uni-valued for $t\in S'$, with equality meaning corresponding elements via the isomorphism of Lemma 3.5.

Proof. Since by $(*1)$ the variation map is well-defined on S, after parallel transporting $[\gamma]$ to S, we have that $\mathrm{var}(\gamma) = \sum_{1\le j\le s} \mathrm{var}\ e_j$; since the map var on functions commutes with var on cycles [15], [14], it follows that var $I^\gamma_f = \sum_{1\le j\le s} \mathrm{var}\ I^{e_j}_f$, as cocycles in $H^1_Z(S,\mathcal{O})$, where $Z := \cup_{1\le j\le s} C_j$. Since by excision $H^1_Z(S,\mathcal{O}) \cong \oplus_{1\le j\le s} H^1_{C_j}(S,\mathcal{O})$, the same argument as in Lemma 3.5 finishes the proof, since var $= \delta : H^0(S,\mathcal{O}) \to H^1_Z(S,\mathcal{O})$ (the Cech coboundary map associates to a section the corresponding cocycle). □

Remark 3.7. We have essentially avoided using holonomic differential systems, by ad-hoc arguments for the case of the Gauss-Manin differential system; this is a prototype for some of the general theory, cf. [17], [20], [21].

The local Laplace transform (at 0) in direction $a = e^{i\theta}$ is described as follows. Let

$$\mathcal{Z}^-_a := \{Z\ |Z \text{ closed homogeneous} \subset \mathbf{C} \text{ s.t. } \langle z,a\rangle < 0, z\in Z\setminus 0\}.$$

The Laplace transform is the map

$$\mathcal{L}^- = (\mathcal{L}^-)_a : \lim_{\to\{Z\in\mathcal{Z}^-_a, r\to 0\}} \mathcal{O}(D_r\setminus Z)/\mathcal{O}(D_r) \longrightarrow (\mathcal{B}^{<1}/\mathcal{B}^{\le -1})_a$$

(cf. notation below for the target space) defined on $h = h(t)$ s.t. $h = \mathrm{var}(\tilde{h})$ (surjectivity of var, cf. [M0]) by

$$\mathcal{L}^-(h)(\tau) := \int_{\sigma_0} e^{-\tau t}\tilde{h}(t)dt + \int_{\delta} e^{-\tau t}h(t)dt,$$

where $\sigma_0 = \partial D_r$ (based at t_0) and δ is a half-line with origin t_0 in direction a.

By [17], the (local) Laplace transform is an isomorphism between the following spaces ai) and bi), $i = 1, 2, 3$ of holomorphic functions on sectors:

a1) no growth conditions (=: sp $\mathcal{O}$, specialization of Sato); a2) sub-exponential growth as $t \to \infty$ (=: $\mathcal{B}^{<1}$); a3) moderate growth as $t \to \infty$ (=: $\mathcal{B}^{\leq 0}$);

b1) $\mathcal{B}^{<1}/\mathcal{B}^{\leq -1}$; b2) $\mathcal{B}^{<1}$; b3) sub-exponential growth as $\tau \to \infty$, moderate growth as $\tau \to 0$ (=:$^{\leq 0}$ $\mathcal{B}^{<1}$).

In the semi-local case, $\mathcal{L}^-_{s.l.}$ is defined by ([PhII])

$$\mathcal{L}^-_{s.l.}(h)(\tau) := \int_{\sigma} e^{-\tau t}\tilde{h}(t)dt + \sum_{1\leq j\leq s}\int_{\delta_j} e^{-\tau t}h(t)dt,$$

where $h = \mathrm{var}\ \tilde{h}$, σ is s.t. $[\sigma] = \sum_{1\leq j\leq s}[\sigma_j]$ (σ_j and δ_j as in the local case above).

Therefore $\mathcal{L}^-_{s.l.}(h)(\tau) = \sum_{1\leq j\leq s}\mathcal{L}^-_{s.l.}(h_j)(\tau)$, if $h = \sum_{1\leq j\leq s} h_j$ (in the sense of the isomorphism of lemma 3.5). Replacing in the spaces bi) above < 1 with ≤ 1, we obtain spaces bi$'$), which are codomains for the semi-local Laplace transform.

Proposition 3.8. *Let p, f, γ and a direction $a = e^{i\theta}$ be as in prop. 3.4; let $[\gamma] = \sum_{1\leq j\leq s}[e_j]$ (as in 1.2).*

If, for $1 \leq j \leq s$, $\mathcal{L}^-_{s.l.}(I^{e_j}_f)$ and $(\mathcal{L}^-_{s.l.}(I^{e_j}_{x^\alpha}))_{\alpha\in I}$ are in the space b1) (resp. b2), resp. b3)), then $I^\gamma_f = E^\gamma_f$ mod a function which is entire (resp. entire of sub-exponential growth at infinity, resp. a polynomial).

PROOF. 1) In the local case (at t_j), since (by Thm. 0.2.1) $I^{e_j}_f = E^{e_j}_f$ (on D'_j) mod $\mathcal{O}(D_j)$, we have $\mathcal{L}^-_a(I^{e_j}_f)(\tau) = \mathcal{L}^-_a(E^{e_j}_f)(\tau)$ in $\mathcal{B}^{<1}/\mathcal{B}^{\leq -1}(\Sigma')$, for some (infinite) sector Σ'_j containing direction a. Since $\mathcal{L}^-_a$ is an isomorphism ([17]) between ai) and bi) (i=1,2,3), the assertions in the local case follow (with $\mathcal{O}(\mathbf{C})$ replaced with $\mathcal{O}(D_j)$).

2) In the semi-local case ($S = S_R$, R fixed, sufficiently large) since, as in the proof of prop. 3.4, $I^\gamma_f = \sum I^{e_j}_f$ and $E^\gamma_f = \sum E^{e_j}_f$ mod $\mathcal{O}(S)$, with all terms defined on S' (resp. S'_j), taking their semi-local Laplace transforms (using additivity and a commutative square with equalities mod $\mathcal{O}(S)$), we have that $\mathcal{L}^-_{s.l.}(I^\gamma_f)(\tau) = \mathcal{L}^-_{s.l.}(E^\gamma_f)(\tau)$ for $\tau \in \Sigma' \subset \cap\Sigma_j$, for Σ' a sector, with equality in the space b1$'$.

3) The assertions of the statement of the proposition are now consequences of 2), extendability (by parallel transport of $[\gamma]$) of I^γ_f, E^γ_f, and the fact that

$\mathcal{L}^-_{s.l}$ is injective on ai) $1 \leq i \leq 3$ ([21], the inverse in the semi-local case also given by integration ($\mathcal{L}'^+$, cf. [17]). $\square$

Remarks 3.9. 1) In the proof of Proposition 3.4, we can avoid the local cohomology argument of lemmas 3.5 and 3.6 by using proposition 3.1 and inversion of $\mathcal{L}^-_{s.l.}$; the solutions I^γ_f and E^γ_f extend analytically over S_R (by parallel transport of $[\gamma]$), therefore $\mathcal{L}^-_{s.l.}$ is defined and injective [21] (with target space, see 2.2, $\mathcal{A}^{\leq r}/\mathcal{A}_{\leq -r}$).

2) In the local situation, instead of the spaces in [21], we can use the space $\mathcal{A}$ of functions holomorphic on sectors truncated to a neighborhood of ∞. Then, since the local Laplace transform is an isomorphism [17] to $\mathcal{B}^{\leq 1}/\mathcal{B}^{<-1}$, we obtain the equivalence between the local algorithms in t and in τ.

Proposition 3.10. *Let p, $f \in \mathbf{R}[x_1, ..., x_n]$. Then (with coefficients computed as in Lemma 0.2.2),*

$$\int_{\mathbf{R}^n} e^{i\tau p(x)} f(x)dx \sim \sum_{\alpha \in I, 0 \leq k \leq l} c^l_\alpha (i\tau)^{-l} \int_{\mathbf{R}^n} e^{i\tau p(x)} x^\alpha dx,$$

with $\sim$ meaning asymptotically as $\tau \to \pm\infty$.

Proof. a) The (phase) integrals are well-defined by truncating f with a C^∞_0-function which is identically one in an open set containing all critical points of $p_{\mathbf{C}}$, and $= 0$ outside a neighborhood of its closure, since (e.g. by [15]) such integrals are asymptotically 0 if supp $f \cap \mathcal{C}_p = \emptyset$.

b) By a partition of unity, it is therefore sufficient to prove the local asymptotic equality:

$$\int_{\mathbf{R}^n} e^{i\tau p(x)} f(x)\phi_j(x)dx \sim \sum_{\alpha \in I, 0 \leq l \leq k} c^l_\alpha (i\tau)^{-l} \int_{\mathbf{R}^n} e^{i\tau p(x)} x^\alpha \phi_j(x)dx,$$

where ϕ_j is a C^∞_0-function which is identically one in a neighborhood of x_j and $= 0$ outside a larger neighborhood. We may, by a change of coordinates, assume $x_j = 0$, ($\phi = \phi_j$).

c) At 0, by a theorem of [15], there exists an admissible n-chain $\Gamma = \Gamma^+$ (i.e. in $H_n(X, X^+)$, where $X^+ = \{ \text{ Im } p_{\mathbf{C}} > 0\}$) such that

$$\int_{\mathbf{R}^n} e^{i\tau p(x)} f(x)\phi(x)dx \sim \int_\Gamma e^{i\tau p(z)} f(z)dz,$$

asymptotically as $\tau \to \infty$ (and similarly w.r.t. a Γ^- and $\tau \to -\infty$.)

In the same way, (for the same n-chains)

$$\sum_{\alpha \in I, o \leq k \leq l} c^l_\alpha (i\tau)^{-l} \int_{\mathbf{R}^n} e^{i\tau p(x)} x^\alpha \phi(x)dx \sim \sum_{\alpha \in I, 0 \leq l \leq k} c^l_\alpha (i\tau)^{-l} \int_\Gamma e^{i\tau p(z)} z^\alpha dz,$$

asymptotically as $\tau \to \infty$ (resp. as $\tau \to -\infty$.)

d) By the above, we have reduced the proof to showing

$$\int_\Gamma e^{i\tau p(z)} f(z) dz \sim \sum_{\alpha,l} c_\alpha^l (i\tau)^{-l} \int_\Gamma e^{i\tau p(z)} z^\alpha dz,$$

asymptotically as $\tau \to \infty$ (resp. as $\tau \to -\infty$,) which is the local algorithm (thm. 0.2.1), via the Borel-Laplace transform. □

Proposition 3.11. *Let $p \geq 0$, $p \in \mathbf{R}[x_1, ..., x_n]$ be s.t. its complexified $p_{\mathbf{C}}$ satisfies (*1), let $\mathcal{R} = \mathcal{R}_p \cap \mathbf{R}$ and let $f \in \mathbf{R}[x_1, ..., x_n]$.*

Then $\gamma(t) := \{p = t\}$ defines a class in $H^c_{n-1}(p^{-1}(t))$ s.t. the real integrals (cf. 0.1) $I_f(t)$ and $E_f(t)$ (the latter mod $\mathcal{O}$) are real-analytic on $t > t_+ := \sup \mathcal{R}$ (resp. $t < t_- := \inf \mathcal{R}$).

For $t > t_+$ (resp. $t < t_-$) they satisfy

$$I_f(t) = E_f(t) \tag{21}$$

mod a real-analytic function on $\mathbf{R}$.

Moreover (writing $[\gamma] = \sum_j [e_j]$ as before), if the complex integrals $I_f^{e_j}$ and $(I_{x^\alpha}^{e_j})_{\alpha \in I}$ are of moderate growth at infinity, then the above equality holds mod a (real-valued) polynomial.

The last condition above holds if all the local solutions of the Gauss-Manin system are of moderate growth at infinity.

Proof. (in the case $t > t_+$, the case $t < t_-$ is similar).

a) By (∗1) for p, $[\gamma(t)]$ is well-defined by parallel transportation.

b) By [20], if $c >> 0$ (sufficient, cf. 1.2, c s.t. all critical values of p are in S_c^-), for $t \in S_c^+$ we have

$$H^c_{n-1}(p^{-1}(t)) \cong H_n(\mathbf{C}^n, p^{-1}(S_c^+)) \cong H_n^\Phi(\mathbf{C}^n).$$

Here $S^+ = S^+(\theta)$, $\Phi = \Phi(\theta)$, where θ is a generic direction, arbitrary for now.

c) It is possible to choose θ s.t. the cuts $C_j = C_j(\theta)$ at t_j (abusively, with the same indexing for points in $\mathcal{R}$ as for $\mathcal{R}_p$) in direction θ satisfy

1) S_c^+ intersects the semi-axis $t > 0$ (e.g. if $\theta \neq -\pi/2$);

2) $C_j \cap \mathbf{R} = \mathcal{R}$ (e.g. if $\theta \neq 0$);

3) θ generic in neighborhood of 0 s.t. $(C_j)_j$ are disjoint.

Then (letting $\theta \to 0$) a sector Σ_+ with origin at t_+ and axis $\{t > t_+\}$ is in the domain of definition of $I_f^\gamma(t)$ and $E_f^\gamma(t)$; at the same time, for t in this range, γ admits a decomposition $[\gamma] = \sum[e_j]$ as before (1.2).

d) By Proposition 3.4 for γ, $I_f^\gamma(t) = E_f^\gamma(t)$ mod $\mathcal{O}(\mathbf{C})$, for $t \in \mathbf{C}^*$, equality of univalued holomorphic functions on sectors avoiding $C_j, 1 \leq j \leq s$, in particular on Σ_+.

For $t \in \Sigma_+$, t real (i.e. $t > t_+$), we have by definition $I_f^\gamma(t) = I_f(t)$ and similarly (since defined mod $\mathcal{O}(\mathbf{C})$, univalued) $E_f^\gamma(t) = E_f(t)$. Therefore for $t \in \Sigma_+$, $I_f(t) - E_f(t) = h(t)$, where $h \in \mathcal{O}(\mathbf{C})$; since it is real-valued, h is real-analytic. This proves (21).

e) The last assertion of the proposition follows from Proposition 3.8 (moderate growth conditions and the Laplace transform).□

Remark 3.12. For p real ($p \geq 0$) we have used complex integrals over the real cycle $\gamma(t) = \{p = t\}$ to prove an algorithm between real integrals (= derivative of moments associated with p). Another cycle, γ_1 (isotopic with γ) is used for computing local (at $0 =$ critical point for $p_{\mathbf{C}}$) real asymptotics using complex integrals over γ_1. The latter satisfies $\gamma_1(t) \subset \{\mathrm{Im} p_{\mathbf{C}} = t\}$, therefore $\gamma_1(t) \cap \mathbf{R}^n = \emptyset$, for any $t > 0$, in particular $\gamma \neq \gamma_1$. Indeed, computations for asymptotics at a critical point by the method of the steepest descent, use directions for which Im $p_{\mathbf{C}}$ =const. ($p_{\mathbf{C}}$ = the phase).

In conclusion, we have

Proof of thm. 0.4.1. a) follows from prop. 3.4; b) from the fact that I_f^e and $I_{x^\alpha}^e$ are solutions of $\mathcal{K}_p$ (see 2.1) and by prop. 3.8; c) is prop. 3.11.

4. Examples

We shall abbreviate I_f^γ as I_f and mod $\mathcal{O}(\mathbf{C})$ as mod $\mathcal{O}$.

4.1. Let $n = 2$ and let

$$p(x, y) = p_\lambda(x, y) = \frac{x^3}{3} - \lambda^2 x + y^2, \tag{22}$$

where $\lambda \in \mathbf{C}$ is a parameter, $x, y \in \mathbf{C}$ are the variables. (This is the mini-versal deformation of the A_2 singularity).

Since $\nabla p = (x^2 - \lambda^2, y)$, a monomial $\mathbf{C}$-basis for $\mathbf{C}[x, y]/I_{\nabla p}$ is $\{1, x\}$. For $\lambda \neq 0$ there are two non-degenerate critical points $(\pm\lambda, 0)$, which for $\lambda = 0$ degenerate to a single critical point (0, of Milnor number = 2).

In general $I_p = t \cdot I_0$; we have, for this p,

$$\frac{1}{3} I_{x^3} - \lambda^2 I_x + I_{y^2} = t \cdot I_0. \tag{23}$$

We shall apply the algorithm of section 3 (which shows that I_f can be reduced mod $\mathcal{O}$ to integrals (I_{x^α}) in a basis with indices $\alpha \in I$, $|I| = \mu_T$) to $f = x^3$ and to $f = y^2$.

For $f = x^3$, since $f = f_0 = x(x^2 - \lambda^2) + x\lambda^2$, we have $f_0 = x\lambda^2 + x\partial_x p$, therefore $g^0 = (x, 0)$ and $f_1 = \mathrm{div}\ g^0 = 1$. Hence

$$I_{x^3} = \lambda^2 I_x + \partial_t^{-1} I_0 \text{ mod } \mathcal{O}. \tag{24}$$

Similarly, from $f_0 := y^2 = (1/2)y\partial_y p$, we have

$$I_{y^2} = \frac{1}{2}\partial_t^{-1} I_0 \bmod \mathcal{O}. \tag{25}$$

Consequences:

a) These latter formulas reduce (23) to a (micro-)differential equation in I_x (if I_0 is known):

$$2\lambda^2 I_x = (\frac{5}{2}\partial_t^{-1} - 3t)I_0 \bmod \mathcal{O}.$$

For $\lambda = 0$, this equation has solutions $I_0 = c\ \ t^{-\frac{1}{6}}$, c a constant; comparing, the Picard-Fuchs equations (at the only critical point 0) give the same solutions for I_0. Indeed, note that since p_0 is quasi-homogeneous (of quasi-homogeneity degree = 1), the monodromy matrix is $M = \text{diag}\ (< \nu, \alpha + 1 > -1)_{\alpha \in I}$, where $\nu = (1/3, 1/2)$ is the weight of quasi-homogeneity.

b) We may alternatively reduce (23) to an equation for I_{x^3} in terms of I_0 :

$$-\frac{2}{3}I_{x^3} = (t - \frac{3}{2}\partial_t^{-1})I_0.$$

For $\lambda = 0$, this and the equation for I_0 (at a) above) imply

$$I_{x^3} = \partial_t^{-1} I_0.$$

If $\gamma(t) = \partial\Omega_t$, ($\Omega_t$ relatively compact in $\mathbf{R}^n$) this follows from

$$I_{x^3} = \int_{\gamma(t)} \frac{x^3 dxdy}{dp} = \int_{\gamma(t)} xdy = \int_{\Omega_t} d(xdy) = \text{ vol } (\Omega_t),$$

the last term being $= \partial_t^{-1} I_0$ by the co-area theorem.

Since p_0 is quasi-homogeneous, I_{x^3} can be also computed by a standard ([1]) change of variables (from $p^{-1}(t)$ to $p^{-1}(1)$, e.g.).

c) For λ arbitrary, we note that (24) and (25) imply $I_{x^3-\lambda^2 x - 2y^2} = 0$. This also follows from the fact that the form $(x^3 - \lambda^2 x - 2y^2)dxdy/dp$ is exact. Indeed, we have $(x^3 - \lambda^2 x - 2y^2)dx \wedge dy = dp \wedge d\theta$, for $\theta = xy$.

4.2. Let $p = p_u$ be the polynomial defined by

$$p_u(z) = |z^4| - (z^2 + \overline{z}^2) - 2u|z^2|, \tag{26}$$

where $u \in \mathbf{C}$ is a parameter, $z \in \mathbf{C}$ is the variable.

Rewriting $z = x + iy$ gives

$$p_u(x, y) = (x^2 + y^2)^2 - 2(x^2 - y^2) - 2u(x^2 + y^2).$$

Allowing next x, y to be complex, we are in the case $n = 2$.

a) For u real, $u \neq 0$, the domain

$$\Omega_u := \{z \in \mathbf{C} \mid p_u(z) = -(u-1)^2\} \tag{27}$$

is relatively compact in $\mathbf{C}$ and for $u \neq 0$ it is a *quadrature domain* ([2]), i.e.

$$\partial\Omega_u = |P(z)|^2 - \sum_{0 \leq k < d} |Q_k(z, \overline{z})|^2 = 0, \tag{28}$$

where P, Q_k are polynomials, $d = \deg P$ and $\deg Q_k = k$.

Indeed, in the present case, we can write $\partial\Omega_u = \{Q_u(z, \overline{z}) = 0\}$, where

$$Q_u(z, \overline{z}) = |z^2 - 1| - 2u(|z|^2 + 1) + u^2 \tag{29}$$

is of the form (28); $\partial\Omega_u$ is the boundary of a Hele-Shaw flow ([11]) with two sources at ± 1, which for $0 < u < 1$ is a disjoint union of two disks with centers at ± 1.

For 1-parameter quadrature domains Ω_u, the moments can be computed from the equation of the boundary $Q_u(z, \overline{z}) = 0$ by ([12]):

$$\frac{-\pi Q'}{Q} = \sum_{k,l=0}^{\infty} \frac{a'_{kl}(u)}{z^{k+1}\overline{z}^{l+1}} \tag{30}$$

where Q' denotes derivative w. r. t. the parameter u. The moments (in the $(z, \overline{z})$ writing) $a_{kl}(u)$ are defined by

$$a_{kl}(u) := \int\int_{\Omega_u} z^k \overline{z}^l dA,$$

where $dA = dxdy$ is the area element.

Computing the left-hand term of (30) in this case (using $\zeta = 1/z$ and long division), we obtain

$$a'_{00} = 2\pi, a'_{11} = 2\pi(u+1), a'_{20} = a'_{02} = 2\pi$$

(and besides the vanishing of the moments of order one).

In coordinates (x, y) this gives:

$$I_0 = 2\pi, I_{x^2} = \frac{\pi}{2}(2u+3), I_{y^2} = \frac{\pi}{2}(2u+1),$$

the integrals being taken over $\gamma_u(t) := \partial\Omega_u$, where $t = -(u-1)^2$.

b) On the other hand, we will show that the algorithm of section 3 implies

$$-I_{x^2}(t) \cdot (1+u) + I_{y^2}(t) \cdot (1-u) = (t - \frac{1}{2}\partial_t^{-1})I_0 \bmod \mathcal{O}_{u,t} \tag{31}$$

w.r.t. a $\gamma_u(t)$ as in Thm 0.4.1.

Indeed, from $I_p = tI_0$, we have first

$$I_{(x^2+y^2)^2} - 2I_{x^2}(1+u) + 2I_{y^2}(1-u) = tI_0. \tag{32}$$

We have $\partial_x p = 4x(x^2+y^2-1-u)$, $\partial_y p = 4y(x^2+y^2+1-u)$. For $u \neq \pm 1$, there are 5 non-degenerate critical points : $0, (\pm\sqrt{1+u}, 0)$, $(0, \pm\sqrt{u-1})$; for $u = \pm 1$, the critical point 0 has Milnor number $\mu = 3$.

By the algorithm of section 3 (parametric version, at a regular value of the parameter) for

$$f = f_u := (x^2+y^2)^2 - (1+u)x^2 + (1-u)y^2 = (1/4)x\partial_x p + (1/4)y\partial_y p,$$

we have $g^0 = (1/4)(x,y)$, $f_1 = \mathrm{div}\ g^0 = 1/2$ and hence $I_f = (1/2)\partial_t^{-1} I_0$. Explicitly, this gives

$$I_{(x^2+y^2)^2} - (1+u)I_{x^2} + (1-u)I_{y^2} = \frac{1}{2}\partial_t^{-1} I_0.$$

Subtracting this from (32), we obtain formula (31) of this sub-paragraph.

c) Finally, in order to compare the results at a) and b) above, we take $t = -(u-1)^2$, or equivalently $u = \sqrt{-t} + 1$ in (31) (here $u > 1$ s.t. $u =$ regular parameter; $t < 0$ avoids the critical values of p_u).

We obtain, by b)

$$-I_{x^2}(t)(\sqrt{-t}+2) + I_{y^2}(t)(-\sqrt{-t}) = (t - \frac{1}{2}\partial_t^{-1})I_0(t) \bmod \mathcal{O}_{\sqrt{-t}}. \tag{33}$$

By a), the right-hand term equals πt; the left-hand term is in $\mathcal{O}_{\sqrt{-t}}$, since both $I_{x^2}(t)$, $I_{y^2}(t)$ depend linearly in $\sqrt{-t}$. Therefore a) too implies that equality mod $\mathcal{O}\sqrt{-t}$ (and not exact equality) holds in (33).

4.3. a) Let $n \geq 1$ and let $p \in \mathbf{C}[x]$ $(x = (x_1, ..., x_n))$ be a polynomial satisfying $(*1)$ and such that its critical points $(x_j)_{1\leq j\leq s}$ are non-degenerate; (note that $s = \mu_T$).

Let f be a polynomial of the form

$$f = x^\beta + h, \tag{34}$$

where $h \in I_{\nabla p}$, $\beta \in I$. For the Fourier integral with phase p and amplitude f defined by (notations differs from 1.3)

$$I_f^\Gamma(\tau) := \int_\Gamma e^{i\tau p(x)} f(x)dx,$$

(where $\Gamma = \Gamma^{(n)} \in H_n^\Phi(\mathbf{C}^n)$, $\Phi = \Phi(\theta)$, θ a generic direction in $\mathbf{C}$) we have, by the first step of the algorithm of Section 3,

$$I_f^\gamma(\tau) = I_{x^\beta}^\Gamma(\tau) + \frac{1}{i\tau} I_{f_1}^\Gamma(\tau),$$

for some $f_1 \in \mathbf{C}[x]$.

b) We shall use the method of the stationary phase to show that

$$I_f^\Gamma(\tau) \sim I_{x^\beta}^\Gamma(\tau) \cdot (1 + O(\frac{1}{\tau})), \tag{35}$$

asymptotically as $\tau \to \infty$, τ in a sector in $\mathbf{C}$. (This checks asymptotically the first step of the algorithm of Thm. 0.4.1).

Since (1.2) $\Gamma = \sum_{1 \leq j \leq \mu} \Gamma_j$, it is sufficient to prove the above formula with Γ replaced by Γ_j, in local coordinates x' at x_j. Let $\Gamma_j = m_{\Gamma_j} \cdot \Delta$, where (notations of Section 1) $\Delta \in H_n(S_j, S_j^+)$ is the standard (Lefschetz) generator (x_j is non-degenerate).

By stationary phase (since $e^{\imath\tau p(x_j)} = (1 + O(\frac{1}{\tau}))$, we have (asymptotically as $\tau \to \infty$, τ in a sector in $\mathbf{C}$)

$$I_f^{\Gamma_j}(\tau) \sim \frac{1}{\tau^{n/2}}(1 + O(\frac{1}{\tau})) \cdot C_j \cdot K_n,$$

with constants C_j, K_n defined by

$$C_j = C_j(f, \Gamma_j, p) = f(x_j) \cdot (\text{ Hess } p(x_j))^{-1/2} \cdot m_{\Gamma_j},$$

$$K_n = (2\pi)^{n/2} \cdot e^{i\pi/4}.$$

Since f was assumed of the form (34), writing $f(x) = x^\beta + \sum_{1 \leq k \leq n} g_k(x)\partial_k p(x)$, we note that $f(x_j) = x_j^\beta$. Therefore

$$C_j(f, \Gamma_j, p) = C_j(x^\beta, \Gamma_j, p),$$

and from this (by a similar computation for the right hand term of (35)) we obtain the asymptotic equality of (35).

c) In particular, for $n = 2$, let

$$p(x, y) = \frac{x^{\mu+1}}{\mu + 1} - x + y^2.$$

Since $\nabla p = (x^\mu - 1, 2y)$, the critical points of p are $(\epsilon^j)_{0 \leq j \leq \mu-1}$, where ϵ is a primitive root of order μ of 1.

Letting $f := x^{\mu+1} = x + x(x^\mu - 1)$, $\mu \geq 2$, we are in the situation above. By the algorithm of Section 3, if $[\gamma(t)]$ is parallel transported in $\mathrm{H}_{n-1}(p^{-1}(t))$ (t large in a sector of axis θ), we have

$$I_{x^{\mu+1}}^\gamma = I_x^\gamma + \partial_t^{-1} I_0^\gamma \text{ mod } \mathcal{O}_t.$$

(For $\mu = 2$, we have used this in example 4.1).

By the above computation using the stationary phase formula (for Γ s.t. $\partial\Gamma = \gamma(t_0)$, some t_0), the leading terms as $\tau \to \infty$ of $I^{\Gamma}_{x^{\mu+1}}$ and I^{Γ}_{x} are both equal with

$$\frac{1}{\tau}\pi \cdot e^{i\pi/4}\sqrt{\frac{2}{\mu}} \sum_{0 \leq j \leq \mu-1} \epsilon^{j/2} \cdot m_{\Gamma_j}.$$

d) For $\mu = 2$, $n = 1$, we also have (similarly) the estimate (for a Laplace transform with phase)

$$\int_{\Gamma} e^{-\tau(\frac{x^3}{3}-x)}dx \sim i\sqrt{\frac{\pi}{\tau}} \cdot (1 + O(\frac{1}{\tau})),$$

which is used in computations for asymptotics ([4],[18]) of the Airy function (where directionally only one of the critical points x_j contributes and $\Gamma = \Gamma_j$ is s.t. $m_{\Gamma_j} = 1$).

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