Asymptotics for extremal moments and monodromy of complex singularities

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Abstract

We present a finite algorithm for the computation of any moment of the solution (= the characteristic function of $\{p < 1\}$, with p a polynomial, assumed to have isolated (complex) critical points) of the truncated extremal n-dimensional L-moment problem, linearly in terms of a finite set of generating moments, in the context of dynamic (i.e. time-dependent) moments. We find that a system of such generators is provided by the moments corresponding to a basis for $\mathbf{R}[x_1, ..., x_n]/I_{\nabla_p}$, where I_{∇_p} is the gradient ideal of p. From this, based on the algebraic formalism for asymptotics of the Fourier transform (Malgrange, 1974), we obtain computations for the coefficients of the asymptotic expansions for the moments in terms of the generators and the monodromy of p.

0. Introduction

The n-dimensional moment problem asks for the existence (and uniqueness) of a Borel measure $\mu > 0$ on $\mathbf{R}^n$ (supported in a fixed subset of $\mathbf{R}^n$) with given moment sequence

$$a_\alpha(\mu) = \int x^\alpha d\mu,$$

$\alpha \in \mathbf{N}^n$, and for which these integrals converge absolutely.

For a survey on existence results on various versions of this problem, mostly in terms of non-negativity of associated matrices with entries depending on the moment sequence, as well as for (non)-uniqueness questions, the reader may consult [A-K], [B], [F], [S].

A related problem is the truncated moment problem, which asks for the existence (and uniqueness) of a measure with given finite sequence of moments. From this one may obtain a solution to the initial moment problem by weak convergence of measures.

Based on convexity in arbitrary normed spaces, the extremal truncated L-moment problem is shown to satisfy existence and uniqueness in this abstract setting. In the case of $L^1(a,b)$, for instance, the solution of the extremal L-moment problem, where L is a bound on the functionals on L^1 in the L^∞ norm, is shown to be (assuming $L = 1$) a.e. the characteristic function of a union of intervals.

This applies also to absolutely continuous measures supported in a fixed compact set $K \subset \mathbf{R}^n$ and to the corresponding truncated $L = 1$-moment problem; the set of extremal moments Σ_K of such measures is easily seen to be compact and convex and an argument using support hyperplanes for Σ_K shows that the solution to the extremal truncated (in the first $|\alpha| \leq d$ moments, for instance) $L = 1$-moment problem is unique (since K is compact, by Weierstrass' approximation theorem) and is of the form $\chi_{\{p<0\}\cap K} \cdot dx$, with p a polynomial (of degree $\leq d$) in n variables (here χ is the characteristic function and dx denotes Lebesgue measure in $\mathbf{R}^n$). Conversely, every such measure is a solution of the extremal moment problem.

The retrieval problem (i.e. the question of finding the measure, given the moments) for this particular $n-$moment problem is therefore well-posed, and retrieval algorithms have been found in a number of cases. For $n = 1$, the classical algorithm of [A-K] is expressed in terms of a Hankel matrix in the moments of degree $\leq 2d$, where $d = \deg(p)$; for $n = 2$, the boundary of quadrature domains is retrieved based on an associated moment problem [P1], [P2]; in the general case of n variables, an algorithm based on a Hankel matrix in terms of dynamic (i.e. time-dependent) moments was recently proposed in [P-P].

A common feature of these algorithms is that they use more moments than are theoretically sufficient in order to determine the solution. (In fact, retrieval is possible precisely by expressing sufficiently many higher order moments in terms of initial ones.) It would be desirable then to find an expression of all the moments in terms of the initial (fewer) that are theoretically needed (and the coefficients of p).

In this paper, we find an answer to this question, in the context of dynamic moments as in [P-P], in terms of the monodromy of the polynomial $p : \mathbf{R}^n \to \mathbf{R}$, assuming that p has isolated (complex) critical points, with distribution-valued moments if p changes sign near the critical point (cf. section 1 for the definitions and main results of singularity theory that we shall use).

Our main result is theorem 2 (section 4), in which an expression (near a critical point of p) of any (derivative of a) moment in terms of a set of generating moments can be obtained by a finite algorithm using Groebner bases. The proof is based on complex fiber integrals and (part of) the algebraic formalism of Malgrange [M] for asymptotic expansions of Fourier transforms with phase $p(z)$ (cf. section 2 for precise details). Namely, we prove that a generating set for the derivatives of the moments is $(a'_\alpha(t))_{\alpha\in I}$, where I indexes an $\mathbf{R}$- basis for $\mathbf{R}[x_1, ..., x_n]/(\partial_1 p, ..., \partial_n p)$. It can be found using a Groebner basis for the ideal $(\partial_1 p, ..., \partial_n p)$ (cf. section 3).

The algorithm which expresses any higher (derivative of a) moment in terms of the moments in a generating set as above, is a finite (polynomial) version of an algorithm of Malgrange; the latter expresses the elements of a certain $\mathbf{C}[[\tau^{-1}]]$-module of finite rank ($\mu =$ the Milnor number) in terms of a basis of it. This module is defined in terms of asymptotics as $\tau \to \infty$ of the (formal) Fourier-Laplace transform in the parameter τ, as $\tau \to \infty$.

In proposition 1 (resp. proposition 3) we prove asymptotic (real) versions of the above algorithm. Although proposition 1 (assuming $p \geq 0$) uses Stokes' theorem and the co-area theorem, and follows from theorem 2, we include it to show the similarities and the differences between the real and the complex method. In proposition 3, based on complex fiber integrals, we extend (via relations among coefficients of various expansions via the Fourier transform as in [M]) the (real) asymptotic algorithm (for fractional power terms only) to the case where p changes sign in a neighborhood of a critical point, and the moments are taken in the sense of distributions.

This implies (corollary 1), that the asymptotics as $t \to 0$, ($t > 0$, resp. $t < 0$) of all derivatives of (distribution-valued) dynamic moments are of the same form

$$a'_\beta(t) \sim \sum_{\alpha_0,q} a^\beta_{\alpha_0,q} t^{\alpha_0} log(t)^q + \mathcal{A},$$

with coefficients $\mathrm{a}^\beta_{\alpha_0,q} \in \mathbf{R}$ and $\mathcal{A}$ denoting a power type series in t, where the indices α_0, q range over $\alpha_0 > -1$, $0 \leq q \leq n-1$ and have the property that $e^{2\pi i \alpha_0}$ is an eigenvalue of the monodromy of p of multiplicity $\geq q+1$ (respectively $\geq q$ if $\alpha_0 \in \mathbf{N}$ and $q \neq 0$.)

Applying proposition 3 to the case (corollary 2) where the monodromy operator is semi-simple (this includes, for instance, every quasi-homogeneous polynomial), from the singular (i.e. modulo a term in $\mathcal{A}$) asymptotics of a generating set of moments, it is immediate to write the singular asymptotics of any higher moment.

Finally, in section 5, we illustrate with several examples.

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1. Some complex singularity theory.

As general references for this section, the reader may consult [A], [A-G-L-V]. Let ϕ be a holomorphic function in a neighborhood of $0 \in \mathbf{C}^n$, $n \geq 1$, such that $\phi(0) = 0$. If ϕ has an isolated singularity at 0 (i.e $d\phi(z) = 0$ iff $z = 0$), then there are sufficiently small neighborhhods, X of $0 \in \mathbf{C}^n$, and T of 0 in $\mathbf{C}$ such that $\phi : X' \to T'$ is a C^∞ locally trivial fibration between $X' = X \setminus \phi(0)$ and $T' = T \setminus \{0\}$. Moreover X can be assumed to be a contractible Stein manifold.

In what follows we shall use the generic notation [ ] to denote the class of an element in a certain quotient group; the implicit assertion is that the concepts are well-defined under change of an element with another in the same class.

For $t \in T'$, the Milnor fibre $X(t) = X \cap \phi^{-1}(t)$ is homotopically a bouquet of μ (the Milnor number) $n-1$-dimensional spheres, therefore the homology group $H_m(X(t), \mathbf{C})$ vanishes except if $m = 0$ or $m = n-1$ and is isomorphic with $\mathbf{C}^\mu$ for $m = n-1$.

The action of the canonical generator (obtained by deforming $\partial T'$ inside T') for $\pi_1(T', t_0)$ (t_0 a base point) on this fibration induces an automorphism h of $H_{n-1}(X(t_0), \mathbf{C})$ (respectively $H^{n-1}(X(t_0), \mathbf{C})$), which is called the monodromy operator. (If $n = 1$, one has to replace H^0 with reduced homology to get dimension μ.) The monodromy theorem asserts that the eigenvalues of h are roots of unity and that in the decomposition $h = SU$, with S semi-simple and U unipotent, one has $(U - I)^n = 0$.

There are induced canonical isomorphisms between the homology groups of the fibres over any two points of T'. This allows one to define, for a given cycle $[\gamma(t_0)] \in H_{n-1}(X(t_0), \mathbf{C})$, its (multi-valued) analytic continuation $[\gamma(t)] \in H_{n-1}(X(t), \mathbf{C})$, $t \in T'$. This, regarded as a parallel transport between the fibres, gives rise to a connection D on the dual (cohomology) bundle $H^{n-1}(X', \mathbf{C}) \to T'$ (the Gauss-Manin connection), defined by $D[\omega] = [d\omega/d\phi]$.

For a fixed holomorphic $(n-1)-$ form ω on X, the integral

$$I(t) = \int_{[\gamma(t)]} [\omega] \tag{1}$$

is well-defined as a multivalued holomorphic function on T'. For $\omega_1, ..., \omega_\mu \in H^n(X, \mathbf{C})$ such that their restrictions $[\omega_1/d\phi]_{|t}, ..., [\omega_\mu/d\phi]_{|t}$ form a basis for $H^{n-1}(X(t), \mathbf{C})$, $t \in T'$, and with $[\gamma_1(t)], ..., [\gamma_\mu(t)]$ the dual basis for $\mathrm{H}_{n-1}(X(t), \mathbf{C})$ (obtained from a dual basis at t_0 by analytic continuity), the integrals (where $1 \leq j, k \leq \mu$)

$$I_{jk}(t) = \int_{[\gamma_k(t)]} [\frac{\omega_j}{d\phi}] \tag{2}$$

satisfy $I'(t) = P(t)I(t)$, a matricial system with holomorphic coefficients for $t \in T'$. Since by Stokes, $I'(t) = \int_{[\gamma(t)]} D[\omega]$, the matrix P is also the matrix of the connection D in the basis $([\omega_j])_{1 \leq j \leq \mu}$.

A main result is that the Gauss-Manin connection is regular singular, i.e. the matrix P extends to a meromorphic function with at most a simple pole at 0. It follows that a fundamental system of solutions of the differential system with matrix P is given by $I(t) = \mathcal{M}(t) \exp(C \log t)$, with $\mathcal{M}(t)$ an invertible matrix with meromorphic entries and $\exp(2\pi i C)$ (with constant matrix C) the monodromy matrix in the above basis at t_0.

By general ordinary differential equations and the monodromy theorem, the intgral $I(t) = I_{jk}(t)$ has an (infinite) expansion of the form

$$I(t) \sim \sum_{\alpha_0,q} d_{\alpha_0,q} t^{\alpha_0} log(t)^q, \tag{3}$$

for $t \in$T', t $\to 0$, where the indices are restricted by q integer such that $0 \leq q \leq n-1$, and $\alpha_0 > -1$ such that $e^{2\pi i \alpha_0}$ is an eigenvalue of the monodromy operator, therefore $\alpha_0 \in \mathbf{Q}$. The integral $I(t)$ defined by (1), for ω an $n-1$-holomorphic form, has a similar expansion with $\alpha_0 > 0$ (cf. [M]).

2. Complex differential forms and asymptotics of integrals

In this section we follow closely results of Malgrange [M]. These allow in particular the transfer of information from the coefficients of asymptotics of complex (fiber) integrals, to the real ones, via the asymptotics at infinity of a formal Fourier-Laplace transform.

Let ϕ be a holomorphic function as in section 1. One can obtain a connection at the level of n-forms by extending the Gauss-Manin connection (we use this in theorem 2, section 4) or equivalently by using asymptotics as $\tau \to \infty$ of a Laplace type transform in the parameter τ (in the generalized sense, with a phase ϕ) of the integrals $I(t)$ defined in section 1.

On the space X (with the notations of section 1), let us consider first the sheaf $\mathcal{O}_X$ of holomorphic functions on X at 0, and, for every $m \in \mathbf{N}$, let Ω^m_X be the sheaf of holomorphic $m-$forms on X.

If $\psi \in \Omega^n_X(X)$ is an arbitrary holomorphic $n-$form, then the $(n-1)-$form $\psi/d\phi$, defined on X' by the condition $d\phi \wedge\ (\psi/d\phi) = \psi$, is a *relative form* on X', i.e. $[\psi/d\phi] \in \Omega^{n-1}_{X/T}(X')$; here the sheaves

$$\Omega^m_{X/T} := \Omega^m_X/(\phi \wedge \Omega^{m-1}_X),$$

for every $m \geq 1$, are $\mathcal{O}_X$- and also $\mathcal{O}_T$-modules. Moreover since 0 is an isolated singularity for ϕ, this relative form extends meromorphically at 0. The dual to the fibre at $t \in T'$ of the sheaf of relative forms is $H_{n-1}(X(t), \mathbf{C})$.

Let $\mathcal{O}_{X,0}$ denote the ring of germs of holomorphic functions on X at 0 ($m(\mathcal{O}_{X,0})$ its maximal ideal); for the $\mathcal{O}_{X,0}-$ modules $M = \mathcal{O}_{X,0}$ or $M = \Omega^m_{X,0}$, let $\hat{M}$ denote the completion with respect to $m(\mathcal{O}_{X,0})$.

Let F_1 and G_1 be the finite type $\mathcal{O}_{X,0}-$ modules defined by

$$F_1 = \frac{d\phi \wedge \Omega^{n-1}_{X,0}}{d\phi \wedge d\Omega^{n-2}_{X,0}}, \quad G_1 = \frac{\Omega^n_{X,0}}{d\phi \wedge d\Omega^{n-2}_{X,0}}, \tag{4}$$

and let $\hat{F_1}$, $\hat{G_1}$ be defined in a similar way, replacing Ω with $\hat{\Omega}$.

The ring morphism $\phi^* : \mathcal{O}_{T,0} \to \mathcal{O}_{X,0}$ induces an $\mathcal{O}_{T,0}$ structure on these modules, which in the case of F_1 is also finite type (by a coherence theorem for sheaves ([M],[A]) and besides $G_1/F_1 \cong \Omega^n_{X/T,0}$ is a $\mathbf{C}-$vector space of dimension μ.

The Gauss-Manin connection extends to $n-$forms as a (relative) connection of $\mathcal{O}_{T,0}$ -modules $D_1 : F_1 \to G_1$, defined by $D_1[\psi] = [d(\psi/d\phi)]$, and is bijective.

The $\mathcal{O}_{T,0}$ module $\hat{F_1}$ is isomorphic to several natural topological completions: $\hat{F_1}^x$ = the $m(\mathcal{O}_{X,0})-$ completion of F_1; $\hat{F_1}^t$ = the $m(\mathcal{O}_{T,0})-$ completion of F_1 as $\mathcal{O}_{T,0}-$ module; and $\hat{F_1}^\tau$ = the completion of F_1 with respect to the filtration $(D_1^{-k}(F_1))_{k\in\mathbf{N}}$.

The completion with respect to τ is alternatively described below in terms of asymptotics of oscillating integrals in the parameter τ, as $\tau \to \infty$. Let $T^- = \{t \in T|\mathrm{Re}\ \ t < 0\}$ and $X^- = X \cap \phi^{-1}(T^-)$ and let $[\Gamma] \in H_n(X, X^-)$ (a so-called admissible chain). By the long exact sequence in cohomology, $\partial\Gamma$ is canonically an element $\gamma(t) \in H_{n-1}(X(t), \mathbf{C})$, $t < 0$.

If, for $\tau \to \infty$ a complex parameter, we integrate $e^{\tau\phi}\psi$ (respectively $\psi/d\phi$) over Γ (respectively over $\gamma(t)$), then, using the asymptotic expansion as $t \to 0$ for $\int_{\gamma(t)} \psi/d\phi$ (as in section 1), it turns out (by the continuity lemma 4.5 of [M] for an $n-1$-form ω such that $d\omega = \psi$) that the Laplace transform is well defined asymptotically and that

$$\int_\Gamma e^{\tau\phi}\psi \ \sim \ \int_0^{-\infty} e^{\tau t}dt \int_{\gamma(t)} \frac{\psi}{d\phi}. \tag{5}$$

as $\tau \to \infty$.

Since the Laplace type transform exchanges derivation with multiplication, one can multiply such integrals of $n-$forms ψ with any formal series in $1/\tau$ by iterating the formula

$$\int_\Gamma e^{\tau\phi}\psi = -\tau \int_\Gamma e^{\tau\phi} D_1^{-1}(\psi), \tag{6}$$

where $D_1^{-1}(\psi) = d\phi \wedge d^{-1}(\psi)$; (since ψ is holomorphic of max degree, $d\psi = 0$, therefore a primitive $d^{-1}(\psi)$ exists). This formula can be used to write an expression for $\int_\Gamma e^{\tau\phi} f dx$ for any holomorphic (or formal series) f, as a formal power series in $1/\tau$ in terms of a basis $(e_\alpha = x^\alpha dx)_{\alpha\in I}$, for n-forms: one starts with an expression for $f_0 = f$ of the form $f_0 = \sum_{\alpha\in I} c_\alpha x^\alpha + \sum_{1\le i\le n} g_i \partial_i \phi$ and iterates for $f_1 = \sum_{1\le i\le n} \partial_i g_i$, etc. This shows that $\hat{G_1}$ has a structure of $\mathbf{C}[[\tau^{-\mathbf{1}}]]-$ module, which is free of rank μ (here $\mathbf{C}[[\tau^{-\mathbf{1}}]]-$ denotes the ring of formal power series in τ^{-1} with coefficients in $\mathbf{C}$).

Similarly, the formula

$$\frac{d}{d\tau} \int_\Gamma e^{\tau\phi}\psi = \int_\Gamma e^{\tau\phi}\phi\psi$$

defines a connection ∇_τ by $\nabla_\tau(\psi) = t\psi \otimes d\tau$. The monodromy matrix of ∇_τ turns out (via the completions described above) to be the inverse of the monodromy matrix of the connection D_1 (in particular ∇_τ is regular singular) and it can be computed by a similar algorithm, starting with a division as above, this time for $f_\alpha = x^\alpha \phi$, $\alpha \in I$.

3. A few facts on Groebner bases.

In $\mathbf{R}[x]$, $x = (x_1, ..., x_n)$, the division algorithm for polynomials works, once a monomial order has been fixed (for the general definition see [C-L-O]). We shall work with the *graded lexicographic order*, which is defined by $x^\alpha < x^\beta$ if either $|\alpha| < |\beta|$, or else if $|\alpha| = |\beta|$, then α precedes β in the lexicographic order for which $x_1 > x_2 > ... > x_n$.

In terms of a monomial order, it makes sense to define multidegree, leading term, etc of a polynomial.

The multidegree of a polynomial $p(x) = \sum_\alpha c_\alpha x^\alpha$ is defined by

$$\alpha^0 = \max\{\alpha | c_\alpha \neq 0\}, \tag{7}$$

an element of $\mathbf{N}^n$. (The total degree of a polynomial does not depend on the monomial order and is by definition $\max\{|\alpha| | c_\alpha \neq 0\}$, an element in $\mathbf{N}$.)

The leading monomial of p is x^{α^0}, and the leading term of p is defined by $LT(p) = c_{\alpha^0} x^{\alpha^0}$.

If I is an ideal in $\mathbf{R}[x]$, a Groebner basis $G = (g_1, ..., g_s)$ for I is defined by the condition $\langle LT(I)\rangle = \langle LT(g_1), ..., LT(g_s)\rangle$ (where $\langle LT(I)\rangle$ is by definition the ideal generated by $\{LT(f) | f \in I\}$). It follows that if $G = (g_1, ..., g_s)$ is a Groebner basis for I, then $g_1, ..., g_s$ are generators for I. Further, every ideal has a Groebner basis, and this is computable starting from any set of generators for I, using Buchberger's algorithm, cf. [C-L-O].

The following proposition (a version of p. 227, [C-L-O]) computes an $\mathbf{R}$-basis for $\mathbf{R}[x]/I$, for I a finite codimensional ideal in $\mathbf{R}$[x].

Proposition. *Let $I \subset \mathbf{R}[x]$ be an ideal such that codim$(I) < \infty$ and let $G = (g_1, ..., g_s)$ be a Groebner basis for I w.r.t. a monomial order on $\mathbf{R}[x]$.*

Then for every $f \in \mathbf{R}[x]$, the division algorithm w.r.t. G expresses f in the form

$$f = \sum_\alpha c_\alpha x^\alpha + \sum_{i=1}^s q_i g_i, \tag{8}$$

where $q_i \in \mathbf{R}[x]$, $1 \le i \le s$ are s.t.

$$\mathrm{multideg(q_ig_i) \le multideg(f)},$$

and $c_\alpha \in \mathbf{R}$, with the indices α ranging in the complement of the monomials in $\langle LT(I)\rangle$.

Further, the coefficients c_α are uniquely determined by f and I, and the monomials in the complement of the monomials in $\langle LT(I)\rangle$ form an $\mathbf{R}$-basis for $\mathbf{R}[x]/I$.

4. An algorithm for higher moments.

Let $p = p(x) \in \mathbf{R}[x_1, ..., x_n]$ be a real polynomial in $n \geq 1$ variables, of total degree $d \geq 1$.

Further, let the *gradient ideal* of p be the ideal in $\mathbf{R}[\mathrm{x}_1, ..., x_n]$ defined by

$$I_{\nabla p} = (\partial_1 p, ..., \partial_n p). \tag{9}$$

We shall consider a monomial $\mathbf{R}-$basis $(x^\alpha)_{\alpha \in I}$, for $\mathbf{R}[x_1, ..., x_n]/I_{\nabla p}$ (obtained e.g. by using a Groebner basis for $I_{\nabla p}$); we shall assume $\nu := \mathrm{codim}(I_{\nabla p}) < \infty$ (therefore the indexing set I is finite) throughout.

For f real-analytic in $\mathbf{R}[x_1, ..., x_n]$, the (fibre) integral

$$I_f(t) = I_{f(x)}(t) := \int_{\{p(x)=t\}} \frac{f dx}{dp} \tag{10}$$

is a C^∞ function of t for t (real) in a neighborhood of 0, $t \neq 0$. By standard results, it has an asymptotic expansion as $t \to 0$, $t > 0$, of the form

$$I_{f(x)}(t) \sim \sum_{\alpha_0, q} a_{\alpha_0, q} t^{\alpha_0} log(t)^q, \tag{11}$$

where the coefficients $a_{\alpha_0,q}$ are in $\mathbf{R}$ and the indices are restricted by q integer such that $0 \leq q \leq n-1$, and by $\alpha_0 > -1$, α_0 ranging in a series of rational numbers with a single denominator, independent of f. A similar asymptotic expansion holds for $t < 0$, as $t \to 0$:

$$I_{f(x)}(t) \sim \sum_{\alpha_0, q} b_{\alpha_0, q} (-t)^{\alpha_0} log(-t)^q. \tag{12}$$

We shall use the integrals $I_\alpha = I_{x^\alpha}, \alpha \in I$ to compute higher moments of p in terms of its I-moments.

Before proceeding, let us look at the problem in its general setting. If we assume $p \geq 0$, the domains $\Omega_t := \{p < t\}$ are relatively-compact in $\mathbf{R}^n$ and their *moments* $(a_\beta(t))_{\beta \in \mathbf{N}^n}$ are well-defined for $t > 0$ by the formula

$$a_\beta(t) = \int_{\{p<t\}} x^\beta dx. \tag{13}$$

Since p has local minima inside Ω_t, one needs to consider singularities for p. We shall see below that one can at best obtain asymptotic results as $t \to 0$, with either $t > 0$ or $t < 0$ (so that t will always be in a small neighborhood of 0); it is well-known that in a region where p does not have critical points, the fiber integrals are asymptotically 0. By stratifying the singularities, one can assume that 0 is the only critical value for p; and if moreover the critical points of p are isolated, it is possible to localize (the real fiber integrals) at a critical point using a partition of unity.

The assumption $\text{codim}_{\mathbf{R}}\text{I}_{\nabla p} < \infty$ is equivalent with the fact that the complexified $p(z)$ of p has only isolated critical points ([C-L-O], p.230); in our results on the fiber integrals we assume we have localized at one of the critical points of p.

We shall sometimes drop the assumption $p \geq 0$ and consider moments as distributions, in the sense that, for $t \neq 0$, $a'_f(t)$ is a C^∞ distribution-valued function of t (with the test functions $f \in C_0^\infty(\mathbf{R}^n)$ replacing x^β in (13)).

From the co-area theorem, (which allows us to fiber integrals by dividing with $d\phi$, cf. [A-G-V] p. 216, or [E-G]), since $p \geq 0$ has no critical points in $\overline{\Omega_t}$, for $t > 0$, the following relation between fiber integrals and derivatives of moments holds

$$I_{x^\beta}(t) = a'_\beta(t), \tag{14}$$

for $t > 0$.

The argument of proposition 1 below relies on the co-area theorem, therefore we shall assume $p \geq 0$. Namely, since for $t > t_0 > 0$,

$$\int_{\{t_0<p<t\}} f(x)dx = \int_{t_0}^{t} ds \int_{\{p=s\}} \frac{f(x)dx}{dp}, \tag{15}$$

and since p has no singularities for $t > 0$ (the asymptotic contribution depends on the lower end t_0), by taking derivatives to the right at t_0 in the above formula, we have

$$\frac{d}{dt_0}(\int_{\{t_0<p\}} f(x)dx) \sim -I_f(t_0), \tag{16}$$

asymptotically as $t_0 \to 0$, $t_0 > 0$.

We shall write this as

$$(\frac{d}{dt_0})^{-1} I_f(t_0) \sim -\int_{\{t_0<p\}} f(x)dx. \tag{17}$$

Proposition 1. *Let $p \in \mathbf{R}[x]$ be a polynomial ($n \geq 1$ variables, degree $d > 0$) such that $p \geq 0$, 0 is the only critical point for p, $p(0) = 0$, and codim$(I_{\nabla p}) < \infty$.*

For $t > 0$ (sufficiently small), let $a_\alpha(t)$ be the α-moment of the domain $\{p < t\}$, with $\alpha \in I$ indexing a basis for the $\mathbf{R}$ vector space $\mathbf{R}[x]/I_{\nabla p}$.

Then for every $\beta \in \mathbf{N}^n$, there exists a finite k ($k \leq |\beta|$) such that, for $t > 0$, $t \to 0$,

$$a'_\beta(t) \sim \sum_{l=0}^{k} (\frac{d}{dt})^{-l} \sum_{\alpha \in I} c_\alpha^l a'_\alpha(t). \tag{18}$$

The constants $c_\alpha^l \in \mathbf{R}$ are computed as follows: let $f_0(x) = x^\beta$ and let

$$f_0(x) = \sum_{\alpha \in I} c_\alpha^0 x^\alpha + \sum_{1 \leq i \leq n} g_i^0(x) \partial_i p(x) \tag{19}$$

be obtained by polynomial division (w.r.t. a Groebner basis of the ideal $I_{\nabla p}$, in the graded lexicographic order in $\mathbf{R}[x]$).

For every $m \geq 1$, if we set

$$f_m = \sum_{1\leq i\leq n} \partial_i g_i^{m-1}, \tag{20}$$

then c_α^m and g_i^m are obtained by polynomially dividing f_m by $(\partial_i p)_{1\leq i\leq n}$.

Proof. Since $p \geq 0$, we have $a'_f(t) = I_f(t)$ by (14), so we need only compute I_f.

From the division formula (19) for $f_0 = f$, we have, for $t_0 > 0$,

$$I_f(t_0) = \int_{\{p=t_0\}} \frac{f dx}{dp} = \sum_{\alpha\in I} c_\alpha^0 I_\alpha(t_0) + \sum_{1\leq i\leq n} (-1)^{i-1} \int_{\{p=t_0\}} g_i^0 \hat{dx_i}, \tag{21}$$

(with notation $\hat{dx_i} = dx_1 \wedge ... \wedge dx_{i-1} \wedge dx_{i+1} ... \wedge dx_n$).

By Stokes' theorem this is asymptotically

$$I_f(t_0) \sim \sum_{\alpha\in I} c_\alpha^0 I_\alpha(t_0) - \sum_{1\leq i\leq n} \int_{\{p>t_0\}} \frac{\partial g_i^0}{\partial x_i} dx;$$

by the co-area theorem, this is further, as noted above

$$I_f(t_0) \sim \sum_{\alpha\in I} c_\alpha^0 I_\alpha(t_0) + \big(\frac{d}{dt_0}\big)^{-1} I_{f_1}(t_0), \tag{22}$$

with f_1 defined by (20) for $m = 1$.

Continuing by induction (with f_m as defined in the statement) and using lemma 1 below, we obtain a finite sum

$$a'_f(t) = I_f(t) \sim \sum_{0\leq l\leq k} \big(\frac{d}{dt}\big)^{-l} \big(\sum_{\alpha\in I} c_\alpha^l a'_\alpha(t)\big), \tag{23}$$

for $t > 0$, $t \to 0$. □

Lemma 1. *With the notations and assumptions of proposition 1, we have that $f_m = 0$ for $m > k := |\beta|$.*

Proof. Let $G = (u_j)_{1\leq j\leq s}$ be a Groebner basis for $I_{\nabla p}$ and let

$$u_j = \sum_{1\leq i\leq n} h_{ji} \partial_i p,$$

$1 \leq j \leq s$. For $m \geq 0$, let

$$f_m(x) = \sum_{\alpha \in I} c^m_\alpha x^\alpha + \sum_{1 \leq j \leq s} q^m_j u_j \tag{24}$$

be obtained by division w.r.t. G (as in section 3).

Defining $g^m_i = \sum_{1 \leq j \leq s} q^m_j h_{ji}$ and $f_{m+1} = \sum_{1 \leq i \leq n} \partial_i g^m_i$ (as in proposition 1), we have

$$\sum_{1 \leq j \leq s} q^m_j u_j = \sum_{1 \leq i \leq n} g^m_i \partial_i p. \tag{25}$$

We shall prove that $\deg(f_{m+1}) < \deg(f_m) - 1$ for every m such that $f_{m+1} \neq 0$; from this the assertion of the lemma follows.

Indeed, the division theorem guarantees

$$\text{multideg}(f_m) \geq \max\nolimits_{1 \leq j \leq s} \text{multideg}(q^m_j u_j),$$

therefore (with graded lex order and using (25)), also

$$\deg(f_m) \geq \max\nolimits_j \deg(q^m_j u_j) = \max\nolimits_i \deg(g^m_i \partial_i p) =$$

$$= \max\nolimits_i(\deg(g^m_i) + \deg(\partial_i p)) \geq \max\nolimits_i(\deg(g^m_i) + \min\nolimits_i \deg(\partial_i p)) \geq \max\nolimits_i \deg(g^m_i);$$

the latter inequality follows from the assumption that 0 is an isolated singularity for p, since this implies $\partial_i p \neq 0$, $1 \leq i \leq n$.

On the other hand, obviously

$$\deg(f_{m+1}) \leq \max\nolimits_i \deg(\partial_i g^m_i) = \max\nolimits_i \deg(g^m_i) - 1$$

and the assertion follows. □

Since p is real, branching at a zero loses information on zeros of p, so in the case p not non-negative we shall use complex fiber integrals (theorem 2 below); for the latter, by lemma 4.5 of [M], we can improve the result of proposition 1 and obtain equality instead of asymptotics.

For $\gamma(t)$ obtained by analytic continuation from $\gamma(t_0) \in H_{n-1}(X_{t_0})$ for some $t_0 \in T'$, the (complex) fiber integrals (for p) are defined by

$$I_{f(z)}(t) := \int_{\gamma(t)} \frac{f(z)dz}{dp(z)}, \tag{26}$$

for $t \in T'$, and have asymptotic expansions of the form (3).

Applying a version of the algorithm sketched at the end of section 2, we obtain the following theorem - this implies proposition 1 (the case of $p \geq 0$, whose proof did not use monodromy).

Theorem 2. *With the notations and assumptions of proposition 1 formula* (18) *holds with equality (instead of asymptotics) for the real fiber integrals defined by* (10).

The same equality holds for the complex fiber integrals, (w.r.t a fixed $\gamma(t)$ as above) without the assumption $p \geq 0$, i.e. we have

$$I_{f(z)}(t) = \sum_{l=0}^{k} \big(\frac{d}{dt}\big)^{-l} \sum_{\alpha \in I} c_\alpha^l I_{z^\alpha}(t), \tag{27}$$

for $t \in T'$, where the iterated integrals $(d/dt)^{-l}$ are taken with 0 *constants for integration.*

Proof. We shall prove the assertion for the complex integrals first. Writing

$$f(z) = \sum_{\alpha \in I} c_\alpha^0 z^\alpha + \sum_{1 \leq i \leq n} g_i^0 \partial_i p,$$

from

$$D_1[\sum g_i^0 \partial_i p dz] = D_1[dp \wedge \sum_i (-1)^{i-1} g_i^0 \hat{dz_i}] = d[\sum_i (-1)^{i-1} g_i^0 \hat{dz_i}] = [\text{ div } g^0 dz],$$

we obtain (with equality in the space F_1 defined in section 2),

$$[f(z)dz] = \sum_{\alpha \in I} c_\alpha^0 [z^\alpha dz] + D_1^{-1}[f_1 dz],$$

where $f_1 := \text{div } g$.

Repeating for f_1 defined by (20) for $m = 1$, etc and using lemma 1, we obtain a finite sum

$$[f(z)dz] = \sum_{0 \leq l \leq k} D_1^{-l} [\sum_\alpha c_\alpha^l z^\alpha dz]. \tag{28}$$

Letting $[\chi^l(z)dz] := {D_1}^{-l}[\sum_{\alpha \in I} c_\alpha^l z^\alpha dz]$, the above formula becomes

$$[f(z)dz] = [\sum_{0 \leq l \leq k} \chi^l(z)dz].$$

Let us note next that if two holomorphic forms satisfy $[\chi(z)dz] = [D_1^{-1}(g(z)dz)]$, then

$$\frac{d}{dt} I_\chi(t) = I_g(t), \tag{29}$$

for every $t \in T'$.

Indeed, using Stokes and lemma 4.5 of [M], we have, for an arbitrary holomorphic $(n-1)$-form ω and for $\arg(t_0) = 0$, that

$$\lim_{t_0 \to 0} \int_{\gamma(t_0)} \omega = 0, \tag{30}$$

therefore

$$\int_{\gamma(t_0)} \omega = \int_0^{t_0} dt \int_{\gamma(t)} \frac{d\omega}{dp}. \tag{31}$$

In our case, taking $\omega = d^{-1}(g(z)dz)$, we have

$$I_\chi(t_0) = \int_{\gamma(t_0)} d^{-1}(g(z)dz) = \int_0^{t_0} dt \int_{\gamma(t)} \frac{g(z)dz}{d\phi}, \tag{32}$$

from which (29) is obtained by taking derivatives at t_0 s.t. arg $t_0 = 0$ arbitrary and the equality follows for every $t \in T'$, since the integrals are holomorphic off 0.

Next, iterating for l arbitrary and applying to $[f(z)dz] = [\sum_l \chi^l dz]$, we obtain the second assertion of the proposition.

From this the first assertion follows since if $p \geq 0$, we can deform $\{p(x) = t\}$ to a family $\gamma(t)$ as above for $t \neq 0$, without changing the holomorphic integrals. □

Remark. The notations for $(d/dt)^{-1}$ used in theorem 2 and proposition 1 are compatible because (as in the proof of theorem 2) by lemma 4.5 of [M]

$$\left(\frac{d}{dt_0}\right)^{-1} I_{f(z)}(t_0) = \int_0^{t_0} dt \int_{\gamma(t_0)} \frac{f(z)dz}{d\phi} = \int_{\gamma(t_0)} d^{-1}(f(z)dz),$$

and by Stokes the latter integral is asymptotically

$$-\int_{\{p>t_0\}} f(x)dx,$$

since for $p \geq 0$, we can deform $\gamma(t_0)$ to $\{p(x) = t_0\}$.

In the following lemma we shall prove a few complements to theorem 2; we make a few additional notations: the space F_b is defined as F_1 of section 2 with the restriction that the coefficients of the forms are taken to be the complexified (holomorphic) functions of real polynomials (n variables); by $\mathcal{T}$ we denote the space of tempered functions (to which all integrals $I_{f(z)}(t)$ belong, cf. [M] for details) and by s (resp. s^ν) the space of real-valued sequences with finitely many non-zero entries (resp. the direct sum with ν terms each of which is s).

Lemma 2. *With notations and assumptions as above, we have the following:*

a) the map $\mathcal{I}: F_1 \to \mathcal{T}$, defined by $\mathcal{I}(f(z)) = I_{f(z)}(t)$ is **C***-linear, injective;*

b) the map $\mathcal{C}: \mathbf{R}[x_1, ..., x_n] \to s^\nu$, defined by $\mathcal{C}(f(x)) = ((c_\alpha^l)_{l\geq 0})_{\alpha\in I}$ is a well-defined surjective **R***-linear map which induces an isomorphism $F_b \to s^\nu$ defined by $\mathcal{C}([f(z)dz]) = ((c_\alpha^l)_{l\geq 1})_{\alpha\in I}$.*

Proof. a) If $\psi := f(z)dz \in dp \wedge \Omega_X^{n-1}$ is such that $I_{f(z)}(t)$ is identically 0, then $\omega := f(z)dz/dp$ is a holomorphic $(n-1)-$form on X and for every $\gamma(t)$ a family of $(n-1)-$cycles in X_t, obtained by parallel transportation as in section 1, we have

$$0 = \int_{\gamma(t)} \omega = \frac{d}{dt}^{-1} \int_{\gamma(t)} D(\omega), \tag{33}$$

therefore $\int_{\gamma(t)} D(\omega) = 0$ (by the monodromy action, since it vanishes identically on one, it vanishes) for all such $\gamma(t)$; this implies $D([\omega]) = 0$, and since D is bijective, that $[\omega] = 0$, i.e. that $[\psi] = 0$ in F_1.

Conversely, if $\psi = 0$ in F_1, then $I_{f(z)}(t)$ is identically 0 by Stokes' theorem (since $(\partial\gamma)(t) = \emptyset$).

b) To show $\mathcal{C}$ is well-defined (i.e. that the coefficients (c_α^l) are uniquely determined from f), let first

$$f = \sum_{\alpha\in I} c_\alpha^0 x_\alpha + \sum_{1\leq i\leq n} g_i^0 \partial_i p = \sum_{\alpha\in I} {c'_\alpha}^0 x_\alpha + \sum_{1\leq i\leq n} {g'_i}^0 \partial_i p \tag{34}$$

be two expressions for $f = f_0$, obtained by dividing w.r.t. a Groebner basis for $I_{\nabla p}$ and defining the quotients g, g' (with vectorial notation $g = (g_i)_i$) as in lemma 1.

By the fact that the remainder is uniquely determined by f and $I_{\nabla p}$ (cf. prop. of section 3), we have $c_\alpha^0 = {c'_\alpha}^0$, $\alpha \in I$. Hence also, $\sum_{1\leq i\leq n}(g_i^0 - {g'_i}^0)\partial_i p = 0$. From this we shall prove below that div $g =$div g', from which it will follow that f_1 is determined by f. Continuing by induction, we obtain that f_l and $(c_\alpha^l)_{\alpha\in I}$ are determined by f, for $0 \leq l \leq k$ (with k finite as in lemma 1).

It remains to prove that div $g =$ div g', i.e. that if h is real-valued and such that $\sum_{1\leq i\leq n} h_i \partial_i p = 0$, then div $h = 0$.

Indeed, let $x_0 \neq 0$ be fixed near $0 \in \mathbf{R}^n$ and let $B_r(x_0)$ be a sufficiently small ball, s.t. dp is nowhere 0 inside $B_r(x_0)$. From

$$0 = \sum_{1\leq i\leq n} \int_{\partial B_r(x_0)} h_i \partial_i p \, \frac{dx}{dp} = \sum_{1\leq i\leq n} (-1)^{i-1} \int_{\partial B_r(x_0)} h_i \hat{dx_i}, \tag{35}$$

we obtain by Stokes

$$\int_{B_r(x_0)} \text{div } h(x) dx = 0,$$

therefore since h is real, letting $r \to 0$, we obtain div $h(x_0) = 0$, and since $x_0 \neq 0$ was arbitrary, div h is identically 0, which proves that $\mathcal{C}$ is well-defined.

The $\mathbf{R}$-linearity of $\mathcal{C}$ follows by the uniqueness of the remainder and the linearity of divergence.

To prove $\mathcal{C}$ surjective, given $((c_\alpha^l)_l)_{\alpha\in I} \in s^\mu$ (with the μ sequences bounded by say k), we define $f_k = \sum_{\alpha\in I} c_\alpha^k x^\alpha$ and solve $f_k =$ div g^{k-1} for g^{k-1} polynomial (e.g. by choosing i and setting $g_i^{k-1} = \int_0^{x_i} f_k(x)dx_i$, $g_j^{k-1} = 0$, for $j \neq i$.) Continuing by induction, for every $m < k$, given g^m a polynomial, we define $f_m = \sum_{\alpha\in I} c_\alpha^m x^\alpha + \sum_{1\leq i\leq n} g_i^m \partial_i p$ and solve $f_m =$ div g^{m-1}, until we reach $m = 0$, wich defines $f = f_0$, a polynomial; using the uniqueness of the remainder, it is easy to see that f satisfies $\mathcal{C}(f) = ((c_\alpha^l)_l)_\alpha$.

In order to show that $\mathcal{C}$ induces a well-defined map on F_b, let $f(z)dz \in dp \wedge d\Omega_X^{n-2}$, with $f(x)$ a real polynomial. We need to show that $(c_\alpha^l) = 0$. For this, by induction it suffices to show that 1) $(c_\alpha^0) = 0$ and 2) $f_1(z)dz \in dp \wedge d\Omega_X^{n-2}$.

1) Since the polynomial $f(z)$ is in the gradient ideal of $p(z)$ in $\mathbf{C}\{z_1, ..., z_n\}$, truncating all the Taylor series in a corresponding expression for $f(z)$ we have that $f(z)$ is in $I_{\nabla p(z)} \cdot \mathbf{C}[z]$, and taking reals part at $z = x$ real, that $f(x)$ is in $I_{\nabla p} \subset \mathbf{R}[x]$, therefore the remainder is 0.

2) Let $f(z) = \sum_i g_i(z)\partial_i p(z) = dp \wedge d\eta$, with η holomorphic. Writing

$$(\sum_i g_i \partial_i p)dz = \sum_i (-1)^{i-1} g_i \partial_i p \; dz_i \; d\hat{z}_i = dp \wedge (\sum_i (-1)^{i-1} g_i d\hat{z}_i),$$

we obtain

$$D_1[dp \wedge (\sum_i (-1)^{i-1} g_i d\hat{z}_i)] = D_1[dp \wedge d\eta] = [d^2\eta] = 0,$$

therefore

$$0 = D_1[dp \wedge (\sum_i (-1)^{i-1} g_i d\hat{z}_i)] = d[\sum_i (-1)^{i-1} g_i d\hat{z}_i] = [\text{ div } g dz],$$

so that $[f_1 dz] = [\text{ div } (g)(z)dz] = 0$.

Finally, to show injectivity, let $\mathcal{C}(f) = 0$, i.e. $(c_\alpha^l) = 0$; by theorem 2, $I_{f(z)}(t) = 0$, and by a) $f(z)dz \in dp \wedge d\Omega^{n-2}$. $\square$

The next proposition is a version of proposition 1, in terms of asymptotics for the real integrals of a not necessarily non-negative polynomial p; the proof is based on theorem 2 (which uses complex integrals) and on the following relations among the coefficients d_{α_0} and a_{α_0}, b_{α_0}, (taking $q = 0$) where $\alpha_0 > -1$, $\alpha_0 \in \mathbf{Q}$, from the asymptotic expansion (3) (resp. (11) or (12)) of $I_{f(z)}(t)$ (resp. $I_{f(x)}(t)$ as $t > 0$, or $t < 0$), as $t \to 0$

$$d_{\alpha_0} = a_{\alpha_0} + b_{\alpha_0} \cdot (-1)^{\alpha_0}, \tag{36}$$

from which it follows that if $\alpha_0 \notin \mathbf{Z}$, the coefficients of the real asymptotics (from each side of 0) can be obtained from the coefficients of the complex asymptotics.

The above relations among coefficients between the asymptotics of complex /real integrals are obtained using theorem 7.6 of [M] and standard formulas for asymptotics of $\int_{\mathbf{R}} e^{i\tau t}\theta(t)t^{\alpha_0}(\log t)^q dt$ (where θ is a C_0^∞ function, identically one on a neighborhood of 0).

Let the inverse (formal) Laplace -Fourier transform of density $f(z)$ and phase $\phi(z) := ip(z)$ (resp. $p(z)$) be denoted by $\mathcal{L}^{-1}$ (resp. $\mathcal{F}^{-1}$), where (as in section 2) equality means well-defined asymptotically as $\tau \to \infty$:

$$\mathcal{L}^{-1}(f)(\tau) = \int_\Gamma e^{\tau p(z)} f(z)dz,\ \mathcal{F}^{-1}(f)(\tau) = \int_\Gamma e^{i\tau p(z)} f(z)dz. \tag{37}$$

Here Γ is an admissible chain for the holomorphic function $\phi(z)$ (as in section 2).

The (complex) fiber integrals (for p) defined by (26) are taken over $\gamma(t)$, $t \in T'$, obtained by analytic continuation from $\gamma(-1) = \partial\Gamma$ (assuming $-1 \in T'$).

Proposition 3. *With the notations and assumptions of theorem 2, if* $f(x) \in \mathbf{R}[x]$*, the fractional power asymptotics of* $I_{f(x)}(t)$ *can be determined from the fractional power asymptotics of* $I_{x^\alpha}(t)$*, using* (36) *and* (18)*, in the sense of asymptotics for the corresponding distributions, as* $t \to 0$*,* t *real, from either side of* 0.

Proof. By theorem 2, we have (for $\gamma(t)$ to be specified)

$$I_{f(z)}(t) = \sum_{0\le l\le k} (\frac{d}{dt})^{-l} \sum_{\alpha\in I} c_\alpha^l I_{z^\alpha}(t), \tag{38}$$

for real t.

From this, the equality for the fractional power part of the asymptotics of the corresponding real fibre integrals as $t \to 0$, $t > 0$ (resp. $t < 0$), is determined by taking the 1-variable Fourier transform of $I_{f(z)}(t)$ as follows.

By theorem (7.6) of [M], there exists an admissible Γ (defining $\gamma(t)$) such that for $\tau \to \infty$,

$$\int_\Gamma e^{i\tau p(z)} f(z)dz \sim \int_{\mathbf{R}^n} e^{i\tau p(x)} f(x)dx. \tag{39}$$

Since, again as $\tau \to \infty$,

$$\int_\Gamma e^{i\tau p(z)} f(z)dz \sim \frac{1}{i}\int_0^{-\infty} e^{i\tau t}dt \cdot I_{f(z)}(t), \tag{40}$$

we have the following relation between complex and real fiber integrals

$$\frac{1}{i}\int_0^{-\infty} e^{i\tau t}dt \cdot I_{f(z)}(t) \sim \int_{\mathbf{R}^n} e^{i\tau p(x)} f(x)dx \sim \int_{-\infty}^{\infty} e^{i\tau t}\theta(t) I_{f(x)}(t)dt,$$

(41)

for $\tau \to \infty$. Here θ is a C_0^∞ function, identically one on a neighborhood of 0, and the last asymptotic equality is obtained integrating term by term the two asymptotic expansions for the real fibre integral; conversely, this determines by (36) the fractional power terms in the asymptotic expansions for $I_{f(x)}(t)$ as $t \to 0$, from both sides of 0. □

Since (again assuming 0 is the only critical point for $p(z)$) more general relations ([M]) with $q \neq 0$ hold among the coefficients $d_{\alpha_0,q}$ $a_{\alpha_0,q}$ and $b_{\alpha_0,q}$ (which, if $\alpha \in \mathbf{Z}$ end recursively in $q = 0$ and cannot be computed by the algorithm of prop. 1), we have that if $a_{\alpha_0,q} \neq 0$ (resp. $b_{\alpha_0,q} \neq 0$), where $\alpha_0 \notin \mathbf{Z}$ or $q \neq 0$, then $a_{\alpha_0,q}$ (resp. $b_{\alpha_0,q}$) can be determined from the coefficients $d_{\alpha_0,q}$; moreover

$$e^{2\pi i\alpha_0} \text{ is an eigenvalue of the monodromy operator of } p, \text{ of}$$

$$\text{multiplicity} \geq q+1, (\text{and such that } \geq q \text{ if } \alpha_0 \in \mathbf{Z} \text{ and } q \neq 0). \qquad (42)$$

From this, we have the following corollary (the form of the real asymptotic expansion follows from [M], essentially by an inverse Fourier transform; to the author's knowledge it has not been made explicit in the context of moments).

Corollary 1. *If p is a polynomial with an isolated (complex) singularity at 0, then the asymptotic expansions as $t \to 0$, $t > 0$ (resp. $t < 0$), of all the moment derivatives $(a'_{x^\beta})_{\beta \in \mathbf{N}^n}$ have terms of the form (up to coefficients): either non-negative integer powers of t, or $t^{\alpha_0} log(t)^q$, where (α_0, q) is as in (42) above (in terms of eigenvalues of the monodromy of p).*

The algorithm of proposition 3 can be applied to compute (part of) the coefficients of the asymptotics of higher moments $a'_f(t)$ (in the sense of distributions if p changes sign near 0), in terms of asymptotics of moments in a basis.

To illustrate, let us consider the case where the singular part, i.e. modulo a power type series term (we shall write this mod $\mathcal{A}$ for short), of the asymptotics has only fractional powers; i.e., let us assume that the monodromy operator is semi-simple.

Then we can write, with Λ denoting the set of eigenvalues of the monodromy of p and coefficients $d_{\alpha_0} \in \mathbf{C}$,

$$I_{f(z)}(t) \sim \sum_{exp(2\pi i\alpha_0) \in \Lambda \setminus \{1\}} d_{\alpha_0} t^{\alpha_0} \text{ (mod } \mathcal{A}). \qquad (43)$$

Let the basis moments I_α,$\alpha \in I$ have the asymptotic expansions

$$I_\alpha(t) \sim \sum_{exp(2\pi i\alpha_0) \in \Lambda \setminus \{1\}} d^\alpha_{\alpha_0} t^{\alpha_0} \text{ (mod } \mathcal{A}). \qquad (44)$$

In formula (27) for $I_{f(z)}(t)$, $t \in T'$, for $0 \leq l \leq k$ let

$$h_l(t) \sim \sum_{exp(2\pi i \alpha_0) \in \Lambda \setminus \{1\}} h^l_{\alpha_0} t^{\alpha_0} \ (\mathrm{mod}\ \mathcal{A})$$

be the asymptotic expansion for $h_l(t) := \sum_{\alpha \in I} c^l_\alpha I_{z^\alpha}(t)$ as $t \to 0$, $t \in T'$. Identifying coefficients, we obtain

$$h^l_{\alpha_0} = (l! C^l_{\alpha_0})^{-1} \sum_{\alpha \in I} c^l_\alpha d^\alpha_{\alpha_0 - l} \tag{45}$$

(here $C^l_{\alpha_0}$ denotes combinations α_0 choose l).

Hence the expansion of $I_{f(z)}(t)$ and the expansions of $I_\alpha(t)$, mod $\mathcal{A}$ have the following relations among their coefficients:

$$d_{\alpha_0} = \sum_{0 \leq l \leq k} (l! C^l_{\alpha_0})^{-1} \sum_{\alpha \in I} c^l_\alpha d^\alpha_{\alpha_0 - l}, \tag{46}$$

for all α_0. From this we obtain similar formulas (since for every $\alpha_0 \notin \mathbf{Z}$, $(-1)^{\alpha_0}$ is not real) for the coefficients a_{α_0}, b_{α_0} for the asymptotic expansions of the real fiber integrals. This proves the following (with indices α_0 depending on the monodromy of p as in (42) above).

Corollary 2. *If p is a polynomial having 0 for its only (complex) critical point and such that the monodromy operator is semi-simple, then the fractional power part of the asymptotic expansion as $t \to 0$, $t > 0$ (resp. $t < 0$, with b_{α_0} replacing a_{α_0}), of the moment a_{x^β}, β arbitrary $\in \mathbf{N}^n$, is of the form*

$$\sum_{e^{2\pi i \alpha_0} \in \Lambda \setminus \{1\}} a^\beta_{\alpha_0} t^{\alpha_0},$$

with (real) coefficients given by

$$a^\beta_{\alpha_0} = \sum_{0 \leq l \leq k} (l! C^l_{\alpha_0})^{-1} \sum_{\alpha \in I} c^l_\alpha a^\alpha_{\alpha_0 - l}, \tag{47}$$

where $a^\alpha_{\alpha_0}$ are the coefficients of the corresponding expansions for the moments a_α, $\alpha \in I$.

If $p \geq 0$, by theorem 2, such relations hold for all the terms of the asymptotic expansions. Recall next that a polynomial is said to be *quasi-homogeneous* if

$$t^{-1} p(x_1, ..., x_n) = p(t^{-\nu_1} x_1, ..., t^{-\nu_n} x_n),$$

identically for $t \neq 0$, where $(\nu_1, ..., \nu_n)$ is the *weight* of p, with $\nu_i > 0, \nu_i \in \mathbf{Q}$. For quasi-homogeneous polynomials with isolated (complex) critical points the monodromy is easily computed (cf. section 2) and in terms of moments, it implies

$$a'_\alpha(t)/a_\alpha(t) = \langle \nu, \alpha + 1\rangle \cdot 1/t, \tag{48}$$

for every $\alpha \in I$ (alternatively by a change of variables also for $\alpha \in \mathbf{N}^n$).

Therefore the monodromy of a quasi-homogeneous polynomial is semi-simple and corollary 2 applies (if p has only isolated critical points).

5. Examples

We shall look at a few examples (only, since in general the integrals are not computable, among the simplest - cubics in 2 variables -are the elliptic integrals) illustrating the algorithm of theorem 2 and the general context (of moments, forms and monodromy).

1. Let the non-negative quadratic polynomial be defined by

$$p(x, y) = x^2 + xy + y^2.$$

Then 0 is an isolated singularity (in $\mathbf{C}$) and $\mu = 1$. We shall check that

$$a'_{(02)}(t) = (2/3) \cdot t a'_0(t).$$

Since $\partial_x p = 2x + y$ and $\partial_y p = 2y + x$, writing

$$f(x, y) := y^2 = -\frac{y}{3}(2x + y) + \frac{2}{3}y(x + 2y)$$

in terms of ∇p, we have $c_0^0 = 0$; further, since $f_1 = 2/3$, we see that the algorithm terminates at $k = 1$; also that $c_0^l = 0$ if $l \neq 1$ and $c_0^1 = 2/3$. Therefore $I_f(t) = h^1(t)$, where $(h^1)' = c_0^1 \cdot I_0(t)$, so that $I_f(t) = 2/3 \cdot a_0(t)$. In terms of derivatives of moments, we have $a'_{(02)}(t) = I_f(t) = 2/3 \cdot a_0(t) = 2/3 \cdot t \cdot a'_0(t)$ (the latter equality by quasi-homogeneity).

Note that for quadratic non-negative polynomials, a Fourier transform computes all the moments simultaneously [H].

2. Let p be a non-negative polynomial (on n variables) satisfying $p(0) = 0$, and such that p has an isolated singularity at 0.

Then for any $1 \leq i \leq n$, taking $f = x_i \partial_i p$, it is immediate (by theorem 2) that $a'_f(t) = a_0(t)$. If moreover p is quasi-homogeneous of weight ν, this implies (as in example 1)

$$a'_f(t) = (1/\langle \nu, 1\rangle) \cdot t \cdot a'_0(t). \tag{49}$$

For instance, the homogeneous polynomials $p_\rho(x, y) = (x^2 + y^2)((1 + \rho)x^2 + y^2)$ have 0 as complex isolated singularity; as $\rho \to 0$, the polynomial $p(x, y) = (x^2 + y^2)^2$, has a non-isolated complex singularity at 0 (isolated over $\mathbf{R}$). The relation (49) holds in both cases, with $\nu = (1/2, 1/2)$.

3. Let $p(x,y) = x^2y^2 + x^2 + y^2$, a non-homogeneous polynomial; then $p \geq 0$, and 0 is an isolated singularity for the complexified of p. For any $s \geq 0$, let us consider

$$f_s(x,y) = xy(x^2+1)^s(y^2+1)^s.$$

Note first that by Stokes $a'_{xy} = 0$; indeed the class of the holomorphic form $xydxdy$ in $\hat{G}_1$ is 0; indeed, by

$$d\bigg(\frac{xy \cdot dxdy}{dp}\bigg) = d\bigg(\frac{ydy}{2(1+y^2)}\bigg) = 0,$$

it is closed, therefore exact since X is Stein.

From $p+1 = (x^2+1)(y^2+1)$, we have the obvious relation

$$(t+1)^s a'_{xy} = a'_{f_s}, \tag{50}$$

therefore $a'_{f_s} = 0$. This checks also by theorem 2, which gives

$$a'_{f_s}(t) = s!\big(\frac{d}{dt}\big)^{-s} a'_{xy}(t). \tag{51}$$

On the other hand, relation (50) is the pull-back via the map $\sigma(x,y) = (x^2,y^2) = (u,v)$ of the following relation (which is immediate by Stokes and quasi-homogeneity) among non-zero moments for the polynomial $p(u,v) = uv$

$$a'_{ss}(t) = t^s a'_0(t),$$

in the sense of distributions, as p changes sign near 0.

4. If $p \geq 0$, the following relations among derivatives of dynamic moments are obvious using fiber integrals, for $\beta \in \mathbf{N}^n$

$$I_{px^\beta}(t) = t \cdot I_\beta(t).$$

Writing $p(x) = \sum_{|\alpha|\leq d} c_\alpha x^\alpha$, by linearity we have, for $\beta \in \mathbf{N}^n$,

$$\sum_{|\alpha|\leq d, \alpha\neq 0} c_\alpha \cdot I_{x^{\alpha+\beta}}(t) + (c_0 - t) I_{x^\beta}(t) = 0.$$

These relations are used for retrieval in [P-P], where a form of the real Nullstellensatz and positivity guarantee uniqueness of the coefficients $(c_\alpha)_\alpha$ of p satisfying them.

By the asymptotics formula of corollary 2, the expressions for arbitrary moments in terms of generators have to be more complicated in general.

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