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A skew normal dilation on the numerical range of an operator

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Abstract. Simple facts about the Poincaré-Neumann double layer potential are used in the construction of a normal dilation, on the numerical range of an arbitrary Hilbert space operator. Recent and old ideas in the theory of the numerical range are unified by this framework. A couple of mapping results for the numerical range are derived.

1. Introduction

Let T be a linear bounded operator acting on a complex Hilbert space H. The *numerical range* of T is the set $W(T) = \{\langle Tx, x\rangle, \ \|x\| = 1\}$. According to Hausdorff-Toeplitz theorem, $W(T)$ is a convex set which contains in its closure the spectrum $\sigma(T)$ of T. In general, the numerical range is not easily computable, even for low rank matrices. On the other hand, estimates of the numerical range of an operator are more accessible than exact spectral computations; for this and other reasons the numerical range is relevant to perturbation theory, numerical analysis, semigroups of contractions, ergodic theory, and possible other fields. The monograph [7] contains many representative results, examples, historical remarks and references on the subject.

Recently, B. and F. Delyon [5] have proved that for any operator T, there exists a constant M with the property that for every polynomial $p(z)$, the estimate:

$$\|p(T)\| \leq M\|p\|_{\infty, W(T)}, \tag{1}$$

holds. This generalizes von Neumann's inequality:

$$\|p(T)\| \leq \sup_{|z|\leq\|T\|} |p(z)|,$$

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and, as the latter, has deep consequences.

For instance, if the numerical range of T is contained in the unit disk, then T is power bounded, that is $\|T^n\| \le M$ for all natural n. Or, starting with (1) one can extend by continuity the functional calculus $f(T)$ to all functions $f \in A(\overline{W(T)})$ which are continuous on $\overline{W(T)}$ and analytic in the interior of $\overline{W(T)}$.

B. and F. Delyon proved this result by an adaptation to the operator valued setting of the double layer potential of a planar domain. They used this new tool for proving a conjecture in ergodic theory attributed to Burkholder. A first aim of this paper is to discuss the link between B. and F. Delyon theorem and some classical aspects of potential theory. For example, we will show that M in the estimate (1) is directly related to C. Neumann's "configuration constant" of a convex domain.

Second, by applying Naimark's dilation theorem we will observe that, for an arbitrary bounded operator T with numerical range contained in the convex set W there exists a normal operator N acting on a larger Hilbert space $K \supset H$ with spectrum on the boundary ∂W, such that, for all analytic functions $f \in A(W)$, one has:

$$f(T)x = 2P[(I + K_W)^{-1} f](N)x, \quad x \in H. \tag{2}$$

Above we have denoted by P the orthogonal projection of K onto H and by $1/2(I + K_W) : C(\partial W) \longrightarrow C(\partial W)$ the double layer (singular integral) operator associated to W. Details about K_W will be provided in Section 2 below.

For instance, in the case $W = \mathbf{D}$ is the unit disk, $K_W f = f(0)$, with the effect that

$$f(T)x = 2Pf(N)x, \quad f(0) = 0, \quad x \in H.$$

Note that in this case N is a unitary operator. This was called by Berger and Stampfli a "skew dilation" of T and was used by them in the proof of certain mapping results for the numerical range, see [2], [3]. Recall that, in contrast, Sz.Nagy's unitary dilation U of a contractive operator T satisfies $f(T) = Pf(U)P$, see [15].

Returning to C.Neumann's double layer potential operator $1/2(I + K_W)$, which was the object of many classical works, see [10], its eigenfunctions $u \in C(\partial W)$ corresponding to the eigenvalue $1/2$ will be interesting for this dilation. These are solutions of the integral equation $K_W u = 0$ and very recently they were studied by Ebenfelt, Khavinson and Shapiro [6]. These authors proved, among other things that, under some regularity restrictions on ∂W, such a function u is characterized by the following matching condition: there are analytic functions $f \in A(W)$ and $g \in A([\mathbf{C} \cup \infty] \setminus W)$, $g(\infty) = 0$, such that

$$u(z) = f(z) = \overline{g(z)}, \quad z \in \partial W. \tag{3}$$

Therefore, returning to the above dilation identity we obtain for a function f as above:

$$f(T) = 2Pf(N)P, \quad K_W f = 0,$$

exactly as in Berger-Stampfli's analysis. We will deduce from this relation a non-trivial estimate for the numerical range of the operator $f(T)$.

Thus, the theory of the numerical range offers one more reason to study the matching problem (3). So far, the only domains Ω for which the operator K_Ω is known to have a non-trivial kernel are the lemniscates $\Omega = \{z \in \mathbf{C}; \quad |r(z)| < 1\}$, where r is a rational function satisfying $r(\infty) = \infty$. Then it is clear that $f = r$ and $g = 1/r$ solve the matching problem, see [6]. By a straightforward generalization of an argument in [6] we will show in the last section that $\ker K_\Omega = 0$ on all domains of the form:

$$\Omega = \{z \in \mathbf{C}; \ |p_d(z)|^2 < |p_{d-1}(z)|^2 + |p_{d-2}(z)|^2 + \ldots |p_0(z)|^2\},$$

where $d > 1$ and p_j are polynomials of exact degree $\deg p_j = j, \ 0 \le j \le d$.

2. Preliminaries

For complete details about the classical potential theory facts recalled in this section, including the rather delicate case of a convex curve with countably many corners, we refer to the excellent note by Schober [13] and Král's lecture notes [10].

Let W be a compact convex set in the complex plane with interior points. Simply the convexity of W insures that its boundary is a rectifiable Jordan curve with one-sided tangents existing at each point, and whose angle with a fixed direction are of bounded variation with respect to arc length. Therefore, with the exception of countably many points, the tangent $\tau(\zeta)$ to ∂W is well defined and continuous as a function of $\zeta \in \partial W$.

For each $z \in \mathbf{C} \setminus \partial W$ fix a determination of $\log(\zeta - z), \ \zeta \in \partial W$, and let $u \in C(\partial W)$ be a real valued continuous function. Its *double layer potential* is the harmonic function:

$$D(u)(z) = \frac{1}{2\pi} \int_{\partial W} u(\zeta) d \arg(\zeta - z).$$

An elementary computation shows that:

$$D(u)(z) = \Re[\frac{1}{2\pi i}\int_{\partial W}\frac{u(\zeta)}{\zeta - z}d\zeta] = \int_{\partial W} u(\zeta)\Re[\frac{d\zeta}{2\pi i(\zeta - z)}].$$

The latter expression gives a second possible definition of $D(u)$ for complex valued u.

Whenever z belongs to the interior of W, respectively to its exterior, we mark the function D by an index $D_i(z)$ respectively $D_e(z)$. It is known that D_i is a continuous function on W and that D_e is continuous on the Riemann sphere $\tilde{\mathbf{C}}$ minus the interior of W. Moreover, at each boundary point $\sigma \in \partial W$ we have representations:

$$D_i(u)(\sigma) = \frac{1}{2}u(\sigma) + \frac{1}{2}K_W(u)(\sigma),$$

$$D_e(u)(\sigma) = -\frac{1}{2}u(\sigma) + \frac{1}{2}K_W(u)(\sigma).$$

Thus, the jumping formula:

$$D_i(u)(\sigma) - D_e(u)(\sigma) = u(\sigma), \quad \sigma \in \partial W,$$

holds. Remark also that for the constant function $u = \mathbf{1}$ we have $D_i(\mathbf{1}) = \mathbf{1}$ and $D_e(\mathbf{1}) = 0$, hence $K_W(\mathbf{1}) = \mathbf{1}$.

The linear continuous transformation $K_W : C(\partial W) \longrightarrow C(\partial W)$ is the classical Neumann-Poincaré singular integral operator (in two real variables); its representing kernel has a simple geometric meaning.

Carl Neumann's approach to the Dirichlet problem $\Delta f = 0$ in int W, $f|_{\partial W} = u$, was essentially the following: solve the equation $1/2(I + K_W)v = u$ and then represent $f = D_i(v)$ as a double layer potential v. Thus, knowing that the operator $I + K_W : C(\partial W) \longrightarrow C(\partial W)$ is invertible solves the Dirichlet problem for an arbitrary convex domain. Later this idea was applied by Poincaré, Fredholm, Carleman to more general classes of domains, in any number of variables. For more historical comments see [10] and [11].

Since K_W leaves the constant functions invariant, it induces a linear operator on the quotient $C(W)/\mathbf{C1}$. A convenient norm for this quotient space is, for u real valued, the oscillation:

$$\text{osc}\ u = \max u - \min u.$$

The following result establishes the invertibility of $I + K_W$ in most of the cases.

Theorem 1. *(C.Neumann's Lemma) For any planar compact convex domain W different from a triangle or a quadrilateral there exists a constant $q < 1$ with the property that:*

$$\text{osc } K_W(u) \le q \text{ osc } u, \tag{4}$$

for every real valued function $u \in C(W)$.

Indeed, under the assumption of the theorem, the Neumann series $\sum_{n=0}^{\infty}(-1)^n K_W^n$ converges to $(I + K_W)^{-1}$ in $C(W)/\mathbf{C1}$. The case of a triangle or a quadrilateral is indeed special, because the optimal constant in (4) is indeed 1. However, Lebesgue has later remarked that in these two special cases K_W^2 is strictly contractive in the oscillation norm; and this suffices for the convergence of the Neumann series.

A complete proof, in modern language, of C.Neumann's lemma appears in [13]. We will call, after C.Neumann, the constant q in the above statement, the *configuration constant* of the convex domain. Various estimates are known for q; for instance, if the boundary of W is of class C^2, then it is easy to show that:

$$q \le 1 - \frac{L}{2\pi R},$$

where L is the length of ∂W and R is the maximum radius of curvature of ∂W. For instance, if W is a disk, then $q = 0$. Or, if W is an ellipse of eccentricity e, $e \le 1$, then $q \le 1 - \frac{1+e}{2}\sqrt{1-e^2}$.

3. The dilation

Let W be a compact convex subset of $\mathbf{C}$ with interior points. Assume first that T is a linear bounded operator on a Hilbert space H with numerical range contained in the interior of W. For a boundary point $z \in W$ we consider a tangent vector t in the positive direction, that is the real line $z + \lambda t$, $\lambda \in \mathbf{R}$, leaves W on the left side (as λ increases). Thus, for every point $w \in W$ we have:

$$\Im \frac{w - z}{t} \ge 0,$$

or equivalently,

$$\Re[it(w-z)^{-1}] \ge 0.$$

According to von Neumann's inequality, see for instance [15], we obtain:

$$\Re[it(T-z)^{-1}] \ge 0,$$

where the inequality is understood in the sense of self-adjoint operators.

Let $\phi : [0,1] \longrightarrow \partial W$ be the arc length parametrization of the boundary of W. It is a continuous function, differentiable on the complement of a countable set, and which admits lateral derivatives $\phi'_\pm$ at each point. Therefore the measure

$$\Re[\frac{1}{2\pi i}\phi'_-(s)(\phi(s)-T)^{-1}]ds,$$

is well defined, non-negative and absolutely continuous with respect to the Lebesgue measure ds. In this way one can define without ambiguity the operator valued positive measure:

$$d\mu_T = \Re[\frac{1}{2\pi i}dz(z-T)^{-1}] \tag{5}$$

carried by the rectifiable curve ∂W. Remark that, by Riesz-Dunford functional calculus, $\mu_T(\mathbf{1}) = I$, that is the total mass of μ_T is the identity operator.

Let T be now an arbitrary operator with numerical range included in W, that is possibly touching the boundary of W. Without loss of generality we can assume that $0 \in \mathrm{int}W$. Let $r < 1$ and consider the operator rT so that its numerical range is contained in intW. Arguing as before, there exists a positive operator valued measure μ_{rT}, carried by ∂W and of total mass equal to the identity. We can regard μ_{rT} as a linear functional on the space $C(\partial W, C_1(H))$ on continuous functions with values in the trace-class ideal $C_1(H)$, the predual of the algebra $L(H)$ of all linear bounded operators. Moreover, the norm of the linear functional μ_{rT} is bounded by one. According to Alaoglu's theorem, the family μ_{rT}, $0 < r < 1$, is relatively compact in the weak* topology of the dual space. Thus there exists a positive operator valued measure μ_T, carried by ∂W and satisfying:

$$\lim_{r_j \to 1} \int u(z)\langle d\mu_{r_jT}(z)x, x\rangle = \int u(z)\langle d\mu_T(z)x, x\rangle, \tag{6}$$

along a certain sequence r_j, for all vectors $x \in H$ and all continuous functions $u \in C(W)$.

Actually, the above computations, when compared to the double layer potential operator, give more. Namely, for an arbitrary polynomial $p(z)$, we have by definition:

$$(D_ip)(w) = \frac{1}{2}p(w) + \frac{1}{2}K_W(p)(w) =$$

$$\frac{1}{4\pi i}\int_{\partial W}\frac{p(\zeta)d\zeta}{\zeta - w} - \frac{1}{4\pi i}\int_{\partial W}\frac{p(\zeta)d\overline{\zeta}}{\overline{\zeta} - \overline{w}}.$$

Let us introduce now the harmonic functional calculus $u(T) = f(T) + g(T)^*$ for $u(z) = f(z) + \overline{g(z)}$ and any analytic functions f, g defined in a neighbourhood of the spectrum of T. The last computation and the very definition of the measure μ_{rT} yield:

$$(D_i p)(rT) = \int p(z) d\mu_{rT}(z),$$

or equivalently,

$$p(T) = \int 2[(I + K_W)^{-1}(p)](z) d\mu_{rT}(z).$$

By passing now to the limit with $r_j \to 1$ and replacing p by any harmonic function u defined on a neighbourhood of W, we obtain:

$$u(T) = \int_{\partial W} 2[(I + K_W)^{-1}(u)](z) d\mu_T(z).$$

Remark that the left hand side is independent of the sequence r_j and that the set $\{[(I + K_W)^{-1}(f + \overline{g})]; \ \ f, g \in \mathbf{C}[z]\}$ is dense in $C(\partial W)$. Thus the limit μ_T is independent of the chosen sequence r_j and we actually have:

$$\lim_{r \to 1} \int u(z) \langle d\mu_{rT}(z)x, x\rangle = \int u(z) \langle d\mu_T(z)x, x\rangle.$$

Next we extend by continuity the above functional calculus, via the second hand term, to all functions $u \in H(W)$ which are continuous on W and harmonic in the interior of W.

Thus, B. and F. Delyon estimate follows:

$$\|u(T)\| \le 2\|(I + K_W)^{-1}u\|_{\infty, \partial W}, \quad u \in H(W). \tag{7}$$

One step further we will appeal to Naimark's dilation theorem, see for instance [1], Appendix 1, pg. 121. It states that there exists a larger Hilbert space $K \supset H$ and a spectral measure E such that $\mu_T(\sigma) = PE(\sigma)P$ for every Borel measurable set $\sigma \subset \partial W$. We have denoted by P the orthogonal projection of K onto H. Thus we have proved the following theorem.

Theorem 2. *Let $T \in L(H)$ be a bounded linear operator and let W be a compact convex set which contains the numerical range of T. Then there exists a normal operator $N \in L(K)$, acting on a larger space K, with spectrum on the curve ∂W, such that:*

$$u(T) = 2P[(I + K_W)^{-1}(u)](N)P, \quad u \in H(W). \tag{8}$$

Assume that $0 \in \mathrm{int} W$. The dilation identity can equivalently be transformed into the form:

$$\frac{1}{2}f(T) - \lim_{r\to 1}\frac{1}{4\pi i}\int_{\partial W} f(\zeta)(\overline{\zeta} - rT^*)^{-1}d\overline{\zeta} = Pf(N)P, \quad f \in A(W). \quad (9)$$

This shows several things: that the limit above exists in the weak operator topology, and that the operator $Pf(N)P$ is determined by T and the analytic function f. Since the space $A(W)$ plus its complex conjugate is dense in $C(W)$ we find that the dilation in Theorem 2 is unique, provided that it is minimal in the obvious sense that K is generated by the vectors $N^k x$ and $N^{*k}x$ with $x \in H$ and $k \geq 0$.

Third, for a fixed $r < 1$ as before, $0 \in \mathrm{int} W$ and a polynomial $f(z)$, we choose a large open disk D_R centered at zero, of radius R, so that $W \subset D_R$. In virtue of Stokes theorem we have:

$$\frac{1}{4\pi i}\int_{\partial W} f(\zeta)(\overline{\zeta} - rT^*)^{-1}d\overline{\zeta} =$$

$$\frac{1}{4\pi i}\int_{\partial D_R} f(\zeta)(\overline{\zeta} - rT^*)^{-1}d\overline{\zeta} + \frac{1}{2\pi}\int_{D_R\setminus W} f'(\zeta)(\overline{\zeta} - rT^*)^{-1}d\mathrm{Area}(\zeta).$$

The line integral along the circle $|\zeta| = R$ can be evaluated via the substituiton $\overline{\zeta} = R^2/\zeta$ and it yields $-\frac{1}{2}f(0)I$. Therefore, we obtain for the harmonic continuation of the function K_W the functional calculus formula:

$$(K_W f)(T) = \frac{1}{2}f(0)I + \lim_{r\to 1}\frac{1}{2\pi}\int_{D_R\setminus W} f'(\zeta)(\overline{\zeta} - rT^*)^{-1}d\mathrm{Area}(\zeta). \quad (10)$$

It is already interesting that the above limit exists, again in the weak operator topology, and it is independent of the radius R. This is consistent with the exact evaluation $K_W f = \frac{1}{2}f(0)$ in the case W is a disc centered at 0.

Thus, putting together the preceding computations we obtain, in the above principal value sense:

$$Pf(N)P = \frac{1}{2}f(T) + \frac{1}{2}f(0) + \frac{1}{2\pi}\int_{D_R\setminus W} f'(\zeta)(\overline{\zeta} - T^*)^{-1}d\mathrm{Area}(\zeta), \quad (11)$$

for all polynomials f. The remarkable feature of this formula is the positivity of the left hand term $\Re f \mapsto P\Re f(N)P$, in contrast with the lack of positivity, in general, of the map $\Re f \mapsto \Re f(T)$. It is exactly the area integral which restores the positivity of the harmonic functional calculus of T.

To illustrate by a simple standard example the above construction let us consider a nilpotent matrix

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The spectrum of T is $\sigma(T) = \{0\}$, its numerical range is the disc centered at zero of radius $|W(T)| = 1/2$, while $\|T\| = 1$. Thus the dilation N is supported by the circle $|z| = 1/2$, while its unitary dilation has spectrum on the unit torus $|z| = 1$. Relation (11) becomes in this case:

$$Pf(N)P = \frac{1}{2}f(T) + \frac{1}{2}f(0),$$

which holds true whenever W is a disc centered at 0, see [2], [3].

To give a first application of Theorem 2 we derive a mapping theorem for the numerical range which takes into account the convexity assumption on W.

Proposition 1. *Let W be a convex compact set, not a triangle or a quadrilateral, and let $q < 1$ be its configuration constant. For any function $f \in A(W)$ and operator T such that $W(T) \subset W$, the following estimates hold:*

$$\frac{2}{1-q}\min_W \Re f - \frac{1+q}{1-q}\max_W \Re f \le \Re W(f(T)) \le \frac{2}{1-q}\max_W \Re f - \frac{1+q}{1-q}\min_W \Re f. \quad (12)$$

Proof. In this proof we denote $K = K_W$; all minima and maxima are taken on the boundary of the set W. Since the operator $I+K : C(\partial W) \longrightarrow C(\partial W)$ has a positive representing kernel, it is positive. Let $u \in C(\partial W)$ be a real valued function. Then

$$2(I+K)^{-1}[u - \min 2(I+K)^{-1}u] \ge 0$$

implies

$$u \ge \min 2(I+K)^{-1}u.$$

Therefore,

$$\text{osc } u \le \max u - \min 2(I+K)^{-1}u.$$

Since the operator K is q contractive with respect to the oscillation norm, we obtain:

$$\max 2(I+K)^{-1}u - \min 2(I+K)^{-1}u \le \frac{2}{1-q}[\max u - \min 2(I+K)^{-1}u],$$

that is,

$$\max 2(I+K)^{-1}u + \frac{1+q}{1-q}\min 2(I+K)^{-1}u \le \frac{2}{1-q}\max u.$$

By substracting from this inequality the oscillation estimate:

$$\frac{1+q}{1-q}[\max 2(I+K)^{-1}u - \min 2(I+K)^{-1}u] \leq \frac{2(1+q)}{(1-q)^2}[\max u - \min u],$$

we find:

$$\max 2(I+K)^{-1}u \leq \frac{2}{1-q}\max u - \frac{1+q}{1-q}\min u.$$

Let N be the normal dilation of the operator T. Since the spectrum of N is contained on ∂W and the continuous functional calculus of N preserves positivity, we infer from the last estimate the operator inequality:

$$2(I+K)^{-1}u(N) \leq [\frac{2}{1-q}\max u - \frac{1+q}{1-q}\min u]I.$$

By changing u into $-u$ we obtain a similar lower bound for the same operator.

Finally, for the analytic function $f \in A(W)$, we take $u = \Re f$, and by Theorem 2 we obtain:

$$\Re f(T) = P2(I+K)^{-1}u(N)P \leq [\frac{2}{1-q}\max u - \frac{1+q}{1-q}\min u]I,$$

which proves the proposition.

4. The matching problem

Let W be as before, a compact convex set with interior points. If the boundary of W is sufficiently smooth, then Neumann- Poincaré's operator K_W is compact and symmetrizable on $L^2(\partial W, ds)$. Its eigenvalues and eigenfunctions are relevant for (quasi)conformal mapping theory and in general for function theory. We will be interested in this section in the specific eigenvalue problem:

$$K_W u = 0, \;\; u \in C(\partial W). \tag{13}$$

Throughout this section we identify a function $u \in C(\partial W)$ with its continuous harmonic extension $u \in H(W)$.

If the boundary of W is sufficiently regular, the above function u can be continued analytically inside W, say to a function f, and antianalytically outside W, say to a function $\overline{g}$, satisfying $g(\infty) = 0$. Conversely, any solution of the matching problem

$$u = f = \overline{g}, \;\;\; f \in A(W), \;\;\; g \in A(\tilde{\mathbf{C}} \setminus W), \; g(\infty) = 0, \tag{14}$$

satisfies equation (13). This shows in particular that $\ker K_W$ is an algebra under pointwise multiplication, and therefore it is either zero or infinite dimensional. This can be seen from the Cauchy integral representation of the second layer potential, and Sokhotskii-Plemelj formulae, see [6].

As mentioned in the introduction, the only known example of domain W with non-trivial elements in the kernel of the Neumann-Poincaré operator is a lemniscate.

Henceforth, let $u \in \ker K_W$ be a non-trivial solution and let $T \in L(H)$ be a linear operator with numerical range contained in W. Then $u^k \in \ker K_W$ for all $k \geq 1$ and the dilation identity (8) becomes:

$$u(T)^k = 2Pu(N)^k P.$$

Remark that $u(N)$ is a normal operator and $\|u(N)\| \leq \|u\|_{\infty,\partial W}$. We can normalize u so that $\|u\|_{\infty,\partial W} = 1$. Therefore $u(N)$ is a contractive operator, and it admits a unitary dilation $U \in L(K')$. Let Q denote the orthogonal projection of K' onto H. Consequently, we have the identity:

$$u(T)^k = 2QU^k Q. \tag{15}$$

Our aim is to prove that the numerical range of $u(T)$ is contained in the closed unit disk. To this end, we can repeat an idea of Berger and Stampfli [3].

Namely, deduce first that the spectral radius of $u(T)$ is, according to the latter relation, less or equal than one. Let $z, |z| < 1$, be a complex number, and consider the series:

$$(I - zu(T))^{-1} = I + \sum_{k=1}^{\infty} z^k u(T)^k =$$

$$I + 2\sum_{k=1}^{\infty} z^k QU^k Q = Q\frac{I + zU}{I - zU}Q.$$

Let E denote the spectral measure of the unitary operator U. Then we infer from Riesz-Herglotz formula:

$$\Re(I - zu(T))^{-1} = \Re \int_{-\pi}^{\pi} \frac{1 + ze^{it}}{1 - ze^{it}} QE(dt)Q \geq 0.$$

According to von Neumann inequality we finally obtain $|W(u(T))| \leq 1$. By putting together the above remarks, we have proved the following result.

Theorem 3. *Let W be a closed compact set with interior points, and let $u \in H(W)$ be a function annihilated by the Neumann-Poincaré operator, K_W. Then for any operator T with numerical range contained in the set W, we have:*

$$|W(u(T))| \leq \|u\|_{\infty,\partial W}. \tag{16}$$

This result was proved for a disk W by Berger and Stampfli [3]. It is also interesting to remark, that in the case W is the convex lemniscate

$$W = \{z \in \mathbf{C};\ r(z) \leq 1\},$$

and $u = r$, where r is a rational function so that $r(\infty) = \infty$, the estimate (16) becomes

$$|W(r(T))| \leq 1,$$

which is one of Kato's main results in [9].

We end this note in a negative tone, by proving that the condition $\ker K_W \neq 0$ is highly unstable with respect to small changes of W. More specifically, a planar domain Ω is called a *generalized lemniscate* of degree d if:

$$\Omega = \{z \in \mathbf{C};\ |p_d(z)|^2 < |p_{d-1}(z)|^2 + |p_{d-2}(z)|^2 + \dots |p_0(z)|^2\},$$

where p_j are polynomials of exact degree $\deg p_j = j,\ 0 \leq j \leq d$. For instance, if $d = 1$, then Ω is a disk; for $d = 2$, Ω can be a cardioid, a disjoint union of two disks, and so on. Remark that the polynomials p_j in the above definition are mutually independent.

Proposition 2. *Let Ω be a connected generalized lemniscate of degree $d \geq 2$. Then* $\ker K_\Omega = 0$.

Proof. The proof is an adaptation of an argument, based on analytic continuation, from [6].

It was proved in [8] that the defining polynomial $P(z, \overline{z})$ of a generalized lemniscate can be put in the form:

$$P(z, \overline{z}) = |p(z)|^2 (1 - \|(A - z)^{-1} x\|^2),$$

where A is a $d \times d$ complex matrix with cyclic vector x and minimal polynomial p.

The boundary of Ω is real algebraic, therefore the function $z \mapsto \overline{z}$ extends from $\partial\Omega$ to the Riemann sphere a as a d-multivalued algebraic function $S(z)$ satisfying the equation:

$$\langle (A - z)^{-1} x, (A - \overline{S(z)})^{-1} x \rangle = 1,\ z \in \mathbf{C}. \tag{17}$$

Let $a \in \partial\Omega$ be a non-ramification point for S. Since $\|(A - a)^{-1} x\| = 1$, then all local branches S_j of S satisfy

$$\|(A - \overline{S_j(a)})^{-1} a\| \geq 1, \quad 1 \leq j \leq d.$$

Denote $S_1(a) = \overline{a}$, so that $\|(A - \overline{S_1(a)})^{-1} x\| = 1$. Since every other branch has different values $S_j(a) \neq S_1(a)$ we infer that:

$$\|(A - \overline{S_j(a)})^{-1} x\| > 1, \quad 2 \leq j \leq d.$$

Therefore $\overline{S_j(a)} \in \Omega,\ 2 \le j \le d.$

Similarly, for every exterior point b of $\overline{\Omega}$ we find that all branches satisfy $\overline{S_j(b)} \in \Omega,\ 1 \le j \le d.$

If the function $S(z)$ has no ramification points on $\tilde{\mathbf{C}} \setminus \overline{\Omega}$, then Ω is the complement of a quadrature domain (in the terminology of Aharonov and Shapiro) and the statement is assured by Theorem 3.19 of [6].

On the contrary, if the algebraic function $S(z)$ has ramification points in $\tilde{\mathbf{C}} \setminus \overline{\Omega}$, then, by repeating the main idea in the proof of the same Theorem 3.19 of [6], there exists a Jordan arc α starting at and returning to a point $a \in \partial\Omega$, such that $\alpha \setminus \{a\} \subset \tilde{\mathbf{C}} \setminus \overline{\Omega}$, having the property that the analytic continuation of $S_1(z)$ along this arc returns to a at another branch, say $S_2(z)$, with $\overline{S_2(a)}\Omega$.

Assume by contradiction that the matching problem on $\partial\Omega$ has a non-trivial solution: $f \in A(\Omega),\ g \in (\tilde{\mathbf{C}} \setminus \overline{\Omega}),\ g(\infty) = 0,\ f(\zeta) = \overline{g(\zeta)},\ \zeta \in \partial\Omega.$ Hence

$$f(\overline{S_1(\zeta)}) = \overline{g(\zeta)},\ \zeta \in \partial\Omega.$$

Let us now describe with the point ζ the curve α, and after returning to the point a, describe $\partial\Omega$ once. By analytic continuation along this path, the matching condition continues to hold, and becomes:

$$f(\overline{S_2(\zeta)}) = \overline{g(\zeta)},\ \zeta \in \partial\Omega.$$

But now $\overline{S_2(\zeta)},\ \zeta \in \partial\Omega$ remains "trapped" into a compact subset $M \subset \Omega$. Thus, putting together the latter identities, we obtain:

$$\max_{\Omega} |f| = \max_{\Omega} |g| = \max_{M} |f|.$$

By the maximum principle, the function f should be a constant. Then g is a constant, too. But $g(\infty) = 0$, that is f and g are identically zero, a contradiction.

For a full description of the framework to which the above proof belongs we refer to [6].

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