Some remarks on spherical isometries

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Paper dedicated to Daoxing Xia on the occasion of his seventieth birthday

Abstract

Spherical isometries are subnormal tuples of commutative Hilbert space operators which generalize the notion of an isometry to the multivariable case. We review recent classification and invariant subspace results concerning spherical isometries. Then we discuss some pathological examples which show that, in general, even the simplest properties of isometries do not persist in this extension to several variables. Finally we propose a general classification scheme for commutative tuples of operators, with emphasis on mixed commutator ranges, which in particular applies to spherical isometries.

1 Introduction

Let H be a separable infinite–dimensional complex Hilbert space. A bounded linear operator $V \in L(H)$ is called an *isometry* if $V^*V = I$, or equivalently, if $\|Vx\| = \|x\|$ for every vector $x \in H$. A typical example is the unilateral shift operator $U_+ \in L(l^2(\mathbb{N}))$.
The celebrated theorem of Wold and von Neumann states that any isometry V can be uniquely decomposed into an orthogonal direct sum $V = U \oplus U_+^{(m)}$ of a unitary operator and an m–fold direct sum of the unilateral shift where $0 \leq m \leq \infty$ (see [SzF]). A cassical theorem of Bela Szökefalvi-Nagy asserts that any contraction $T \in L(H)$ on a Hilbert space can be realized as the compression of an isometry to a co-invariant subspace ([SzF]). On the other hand, the basic isometry U_+ can be represented as the multiplication by

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the variable z on the Hardy space $H^2(\mathbb{D})$ of the unit disk in the complex plane. Thus any contractive operator comes together with a unique minimal isometric dilation, and the latter can be represented on a vector–valued function space. This fundamental link between the geometry of Hilbert spaces and analytic function theory in the unit disk was the source for many important developments in both fields during the last half century. For details and further references the reader can consult [SzF] and [Ni].

The natural question whether a similar framework exists for commuting systems T of bounded operators has definitely a negative answer. Although a rather refined spectral theory exists (see [T]), with many links to classical function theory, see for references [Ar], [EP], [Ru], the simplicity resulting from the Sz.–Nagy dilation theorem and the related functional model is missing in the context of several commuting operators. Instead, special classes of commuting n-tuples were studied, usually motivated by specific Hilbert–space or function–theoretical questions. Among them are the spherical isometries which form the object of this note.

The aim of the present paper is two fold: to review some known facts about spherical isometries, by providing complete proofs for a basic classification result, and second, to offer some pathological examples of such systems of operators. This class of commuting systems of Hilbert–space operators can be considered as one possible generalization of the notion of an isometry to the multivariable case. A simple and powerful link with the function theory in the unit ball of $\mathbb{C}^n$ exists, some rigidity results à la Wold-von Neumann are known, applications to interpolation problems and systems theory can be established, some invariant–subspace results were recently proved; but in spite of this progress, many pathologies are easily available. Thus the present picture of the theory of spherical isometries is complex, yet unfinished.

Besides spherical isometries, a few notable several variable generalizations of isometries are: tuples of commuting isometries and the related polydisk models for operators, see [BLT], [BV], [CV], [GS]; the non-commutative framework of G. Popescu [Po1]; Cuntz algebras [Cu].

The list at the end of this note contains references to various aspects of spherical isometries, but also to more distant related subjects of function theory and operator theory. Although the bibliography is rather long, it is far from being exhaustive. We apologize for any omissions.

2 Preliminaries

2.1. Let H be a separable infinite–dimensional complex Hilbert space and let $n \geq 1$ be a fixed integer. A system $T = (T_1, T_2, \ldots, T_n)$ of commuting bounded linear operators on H is called a *spherical isometry* if

$$T_1^*T_1 + T_2^*T_2 + \ldots + T_n^*T_n = I. \tag{2.1}$$

The simplest example is furnished by the Lebesgue space $H = L^2(\mu)$ of a positive Borel measure supported by the unit sphere in $\mathbb{C}^n$, and correspondingly by the n-tuple of multiplication operators by the complex variables $z = (z_1, z_2, \ldots, z_n)$, $M = (M_{z_1}, M_{z_2}, \ldots, M_{z_n})$. This system is actually a *spherical unitary*, that is, a spherical isometry $T = (T_1, \ldots, T_n)$ such that all components T_i are normal. Let $P^2(\mu)$ be the closure of all complex polynomials in $L^2(\mu)$, and let T be the restriction of the n-tuple M to this invariant subspace. Then T is a spherical isometry, which in general is not normal. It is well known that any spherical isometry with a joint cyclic vector is of this form.
A more intricate class of examples is obtained from a system of commuting isometries $[V_i, V_j] = 0,\ V_i^* V_i = I,\ 1 \le i, j \le n$. Indeed, in this case

$$T = (\frac{V_1}{\sqrt{n}}, \ldots, \frac{V_n}{\sqrt{n}})$$

is a spherical isometry. Some recent attempts of understanding the classification of commuting isometries are contained in [BV], [DF], [GS], [BerLi].
Thus there are good reasons not to aim at a complete classification of spherical isometries up to joint unitary equivalence or joint similarity. The examples constructed in Section 3 will support and clarify this statement.
However, there are a few positive results which make the study of spherical isometries interesting. We briefly recall some of them, without proofs, in what follows.

2.2. By using Agler's convexity and extension technique A. Athavale has proved the following basic fact, see [At2].

Theorem (Athavale) *Any spherical isometry is subnormal.*

To be more specific, given a spherical isometry T on a Hilbert space H, there exists a Hilbert space K, containing H as a closed subspace, and a spherical unitary N on K which leaves H invariant such that $T = N|H$.
The main idea in Athavale's proof is to lift the ordered functional calculus

$$\Phi : \mathbb{C}[z, \overline{z}] \longrightarrow L(H), \quad \Phi(z^k \overline{z}^l) = T^{*l} T^k, \quad k, l \ge 0,$$

to a star homomorphism

$$\Psi : \mathbb{C}[z, \overline{z}] \longrightarrow L(K), \quad \Psi(z^k \overline{z}^l) = N^{*l} N^k, \quad k, l \ge 0.$$

The basic positivity properties of the linear map Φ, plus the isometry assumption (2.1), make this lifting possible, see [At1], [At2].
In this way, the theory of commutative subnormal tuples of operators is applicable to spherical isometries. In particular, any spherical isometry T can be uniquely decomposed as $T = N \oplus V$ into a normal tuple N with joint spectrum in the unit sphere and a *completely non-normal* or *pure* spherical isometry V.

2.3. Let T be a spherical isometry. Taylor's joint spectrum of T will be denoted by $\sigma(T)$, while $\sigma_{ess}(T)$ will stand for the joint essential spectrum. Similarly, $\sigma_r(T)$ denotes the right spectrum and so on, see [EP] for precise definitions.
It is known that

$$\sigma(N) \subset \sigma_r(T),$$

where N is the minimal normal extension of T, see [Pu1]. Furthermore, it is elementary to check that the spectrum of T is contained in the polynomially convex hull of the spectrum of N. $\sigma(T)$ is contained in all commutative Banach algebra joint spectra $\sigma_B(T)$, $T \in B^n$. The first non-trivial open question we encounter is: under which conditions is $\sigma(T)$ equal to the polynomial convex hull of $\sigma(N)$? For more about the operator-theoretical aspects of polynomial convexity see [AlW].
Let us consider a simplified situation. Namely assume that the finiteness restriction $\mathrm{rank}[T_i^*, T_i] < \infty$, $1 \le i \le n$, is imposed. Then the tuple T is essentially normal, and by known general theory (see for instance [EP]) the set $\sigma(T) \cap \mathbb{B} = (\sigma(T) \setminus \sigma_{ess}(T)) \cap \mathbb{B}$ is a (possibly singular) complex analytic subvariety of the unit ball of $\mathbb{C}^n$.
Although no classification of these spherical isometries with finite rank self–commutators exists, their spectrum has a simple geometric picture, first revealed in the works of D. Xia and J. Pincus [PX]. We state below one of their main results.
First some notation. For a fixed completely non-normal spherical isometry T, the subspace

$$\mathcal{D} = \Big(\sum_{j=1}^{n} [T_j^*, T_j]H\Big)^-$$

will be called the *defect space* of T.
In what follows we assume that $\dim \mathcal{D} < \infty$. Then, as we will see later, $\mathcal{D}$ is an invariant subspace for the operators T_i^* and $[T_i^*, T_j]$, $1 \le i, j \le n$. Let us denote, following [Xia1],

$$\Lambda_i = (T_i^* | \mathcal{D})^*, \quad C_{ij} = [T_i^*, T_j] | \mathcal{D}, \quad 1 \le i, j \le n.$$

Some elementary but tedious computations carried out in [Xia1] show that the spectrum of any minimal normal extension of T is contained in the real algebraic set A on the unit sphere of $\mathbb{C}^n$ given by the system of equations

$$\det[(z_i - \Lambda_i)^*(z_j - \Lambda_j) - C_{ij}] = 0, \quad 1 \le i, j \le n.$$

A dimension count shows then that A is a real algebraic curve, see [PX]. Since the set $\sigma(T) \cap \partial\mathbb{B}$ "bounds" the rest of the spectrum $\sigma(T) \cap \mathbb{B}$, we find that the whole spectrum is contained in a complex algebraic curve. Hence the next theorm.

Theorem (Pincus-Xia) *The joint spectrum of a pure spherical isometry with finite rank self-commutators is contained in the intersection of a complex algebraic curve and the closed unit ball of* $\mathbb{C}^n$.

Thus it is reasonable, although not completely proved, to expect that such a spherical isometry T admits a "uniformization" $T_j = f_j(S), 1 \leq j \leq n$, where $f_1, \ldots, f_n$ are analytic functions of a single variable and S is a single bounded linear operator. A similar spectral picture can be obtained, as for instance in [BV], via M. Livsic theory of operator colligations and their determinatal varieties.
To give a simple example, as a direct byproduct of this spectral analysis one obtains the following generalization of Morrel's theorem, cf. [Mo] and [Xia1].

Theorem (Morrel, Xia) *Let T be a pure spherical isometry with one–dimensional defect space. Then*

$$T \equiv (a_1 U_+ + b_1, a_2 U_+ + b_2, \ldots, a_n U_+ + b_n),$$

where $a = (a_1, \ldots, a_n) \in \mathbb{C}^n \setminus \{0\}$, $b = (b_1, \ldots, b_n) \in \mathbb{C}^n$ satisfy

$$|a|^2 + |b|^2 = 1, \qquad \langle a, b \rangle = 0,$$

and $\equiv$ is the joint unitary equivalence relation.

However, we should remark that the classification of spherical isometries having a two–dimensional defect space is not yet fully understood.
In addition to the above mentioned spectral picture, a trace formula for commutators of functions of T and T^* was established in [Xia2], [PX]. We do not expand here these interesting aspects.

2.4. There is much analogy between the contents of the last paragraph and the recent studies of subnormal operators of finite rank self-commutators, see [McY], [Pu2], [Xia3], [Yak]. We elaborate below a single aspect: a canonical matricial decomposition of spherical isometries possessing finite–dimensional defect space.
Assume as before that the defect space $\mathcal{D}$ is finite dimensional, and that T is a completely non-normal spherical isometry. Let N be a minimal normal extension of T, decomposed with respect to the sum $K = H \oplus (K \ominus H)$

$$N_j = \begin{pmatrix} T_j & U_j \\ 0 & V_j \end{pmatrix}.$$

The commutator identities $[N_j^*, N_k] = [N_j, N_k] = 0$ lead to the following relations, valid for $1 \leq j, k \leq n$,

$$T_j U_k + U_j V_k = T_k U_j + U_k V_j,$$

$$[V_j, V_k] = 0,$$

$$[T_j^*, T_k] = U_k U_j^*,$$

$$T_j^* U_k = U_k V_j^*,$$

$$[V_j^*, V_k] = -U_j^* U_k.$$

The operators U_j turn out to be of finite rank, with ranges spanning the defect space $\mathcal{D}$. The invariance of $\mathcal{D}$ under each operator T_j^* is therefore proved by one of the intertwining relations above.

In analogy to the theory of Jacobi matrices, we consider the finite–dimensional subspaces

$$\mathcal{D}_m = \bigvee_{k_1+\ldots+k_n \leq m} T_1^{k_1} \ldots T_n^{k_n} \mathcal{D}, \quad m \geq 0.$$

A simple recurrence argument based on the commutator inclusion $[T_j^*, T_k]H \subset \mathcal{D}_0 = \mathcal{D}$ shows that, for any $m \geq 0$ and $1 \leq k \leq n$, we have $T_k^* \mathcal{D}_m \subset \mathcal{D}_m$. The complete non-normality of T implies that the Hilbert space H can be decomposed as

$$H = \mathcal{D}_0 \oplus (\mathcal{D}_1 \ominus \mathcal{D}_0) \oplus (\mathcal{D}_2 \ominus \mathcal{D}_1) \oplus \ldots .$$

With respect to this decomposition each operator T_j admits a Jacobi type block-matrix decomposition

$$T_j = \begin{pmatrix} D_1^j & 0 & 0 & 0 & \ldots \\ C_1^j & D_2^j & 0 & 0 & \ldots \\ 0 & C_2^j & D_3^j & 0 & \ldots \\ 0 & 0 & C_3^j & D_4^j & \ldots \\ \vdots & & \vdots & & \ddots \end{pmatrix}.$$

We list in addition some commutation relations deduced from $[T_j, T_k] = 0$ and the fact that $[T_j^*, T_k]$ has range in the first space $\mathcal{D}_0$

$$[D_1^{j*}, D_1^k] + C_1^{j*} C_1^k = U_k U_j^* | \mathcal{D},$$

$$[D_{p+1}^{j*}, D_{p+1}^k] + C_{p+1}^{j*} C_{p+1}^k = C_p^k C_p^{j*},$$

$$C_{p+1}^k C_p^j = C_{p+1}^j C_p^k,$$

valid for $p \geq 1$ and $1 \leq j, k \leq n$.

Once the latter matricial decomposition is established, it is standard operator theory to prove the following classification result (see Theorem 3 in [Xia1]).

Theorem (Xia) *Let T be a pure spherical isometry with finite–dimensional defect space $\mathcal{D}$ and let*

$$\Lambda_i = (T_i^* | \mathcal{D})^*, \quad C_{ij} = [T_i^*, T_j] | \mathcal{D}, \ 1 \leq i, j \leq n.$$

Then the system of matrices (Λ_i, C_{ij}), $1 \leq i \leq j \leq n$, is a complete set of unitary invariants for T.

However, the matrices $(\Lambda_i, C_{ij}),\ 1 \le i \le j \le n$, are not free; they are subject to certain complicated non-linear operatorial equations. In Section 4 we shall prove a more general version of the last theorem valid for arbitrary completely non-normal n-tuples.
The matricial model described in this section shows, from another perspective, the difficulties arising in the construction of spherical isometries with low dimensional defect space. It is worth mentioning that the same matricial structure is encountered in the theory of orthogonal polynomials in several (real) variables, see [Xu1], [Xu2]. Again, this circle of ideas is not expanded in the present paper.

2.5. Let $S \in L(H)$ be a subnormal operator. A famous theorem of Scott Brown says that the invariant-subspace lattice $\mathrm{Lat}(S)$ of S is non-trivial. Using the Scott Brown method Olin and Thomson [OT] proved that each subnormal operator $S \in L(H)$ is *reflexive*, that is, each operator in

$$\mathrm{Alg\ Lat}(S) = \{T \in L(H);\ TM \subset M \text{ for all } M \in \mathrm{Lat}(S)\}$$

belongs to the unital WOT-closed subalgebra $\mathcal{W}_S \subset \mathcal{L}(\mathcal{H})$ generated by S.
A result of K.Yan [Yan] shows that the joint invariant-subspace lattice of a subnormal n-tuple $S \in L(H)^n$ is again non-trivial, but the reflexivity question for subnormal tuples is still open, except in some special cases. For commuting families of isometries a positive answer was given by Bercovici [Ber].

Theorem (Bercovici) *Any commuting tuple $T = (T_1, \dots, T_n) \in L(H)^n$ of isometries on a Hilbert space is reflexive.*

The question whether each spherical isometry $T \in L(H)^n$ is reflexive is open. The answer is positive if the Taylor spectrum of T is dominating for $\mathbb{B}$, that is, if

$$\|f\|_{\infty,\mathbb{B}} = \|f\|_{\infty,\mathbb{B}\cap\sigma(T)} \quad (f \in H^\infty(\mathbb{B})),$$

or if T possesses an isometric w^*-continuous $H^\infty(\mathbb{B})$-functional calculus ([E2]).

Theorem (Eschmeier) *Each subnormal tuple $S \in L(H)^n$ with $\sigma(S) \subset \overline{\mathbb{B}}$ such that*

(i) the Taylor spectrum of S is dominating in $\mathbb{B}$ or

(ii) S possesses an isometric w^*–continuous $H^\infty(\mathbb{B})$–functional calculus

is reflexive.

It was shown by Müller and Ptak [MP] that spherical isometries T are at least *hyporeflexive*, that is, $\mathcal{W}_T = \mathrm{Alg\ Lat}(T) \cap (T)'$ where $(T)'$ denotes the commutant of

T. The known reflexivity proofs for subnormal operators or tuples T depend on the fact that the w^*–closed unital subalgebra $\mathfrak{A}_T$ of $L(H)$ generated by T satisfies suitable factorization properties. Via trace duality the Banach space $\mathfrak{A}_T$ is the norm dual of the quotient space $\mathcal{C}^1(H)/{}^{\perp}\mathfrak{A}_T$. Hence each w^*–continuous linear functional of $\mathfrak{A}_T$ is of the form

$$L = \sum_{i=1}^{\infty} [x_i \otimes y_i]$$

where $(x_i)_{i\geq 1}$ and $(y_i)_{i\geq 1}$ are square–summable sequences in H and $[x \otimes y]$ denotes the equivalence class of the rank–one operator $H \to H$, $\xi \mapsto \langle \xi, y \rangle x$.

Theorem (Eschmeier) *The dual algebra $\mathfrak{A}_T$ generated by a spherical isometry $T \in L(H)^n$ possesses the factorization property $(\mathbb{A}_{1,\chi_0})$, i.e., for each sequence $(L_k)_{k\geq 1}$ of w^*–continuous linear functionals $L_k : \mathfrak{A}_T \to \mathbb{C}$, there are vectors $x \in H$, $y_k \in H$ $(k \geq 1)$ such that $L_k = [x \otimes y_k]$ $(k \geq 1)$.*

A quantitative version of this result giving norm bounds for the vectors x and y_k can be found in [E3] (Corollary 1.10). In the case that $\mathfrak{A}_T \cong H^\infty(\mathbb{B})$ as dual algebras the last result can be used to prove the reflexivity of T. Necessary for the isomorphism $\mathfrak{A}_T \cong H^\infty(\mathbb{B})$ to hold is the existence of w^*–continuous characters for the dual algebra $\mathfrak{A}_T$. However, in the next section we shall see that there are spherical isometries $T \in L(H)^n$ for which the dual algebra $\mathfrak{A}_T$ has no w^*–continuous characters at all.

2.6. There is much to say about the role of spherical isometries, or more generally of spherical contractions, in questions of bounded interpolation, corona type decompositions, commutant lifting on the unit sphere of $\mathbb{C}^n$. Similar questions are currently being investigated for commuting isometries, see [BV], [BT]. Due to space limitation, we will present elsewhere some details about these subjects.

3 Approximation on subsets of the sphere

An isometry $T \in L(H)$ with spectrum contained in the unit circle is obviously a unitary operator. A result of Izzo [I] on the failure of polynomial approximation on polynomially convex subsets of the unit sphere implies that the situation is very different in the multivariable case. For a compact set $K \subset \mathbb{C}^n$, we denote by $\hat{K}$ its polynomially convex hull in $\mathbb{C}^n$.

Theorem 3.1 *For every $n \geq 3$, there is a non–normal spherical isometry $T \in L(H)^n$ such that the polynomially convex hull of $\sigma(T)$ is contained in $\partial\mathbb{B}$.*

Proof. Fix an integer $n \geq 3$. By the cited result of Izzo [I] there is a polynomially convex compact subset K of $\partial\mathbb{B}$ such that the uniform closure $P(K)$ of the polynomials in n variables in $C(K)$ is a strict subalgebra of $C(K)$. By Hahn–Banach's theorem there is a non–zero complex measure μ on K with $\mu \in P(K)^{\perp}$. The tuple

$$T = M_z \in L(P^2(|\mu|))^n$$

is a spherical isometry with Taylor spectrum contained in the polynomially convex hull of $\sigma(M_z, L^2(|\mu|))$. Hence $\sigma(T) \subset K \subset \partial\mathbb{B}$.
Let us assume that T is normal. Then $P^2(|\mu|)$ would be a reducing subspace for $N = M_z \in L(L^2(|\mu|))^n$ and hence $\mathbb{C}[z, \overline{z}] \subset P^2(|\mu|)$. By the Stone–Weierstraß theorem it follows that $P^2(|\mu|) = L^2(|\mu|)$. But then, for $f \in C(K)$ arbitrary, we could choose a sequence (p_k) of polynomials with limit f in $L^2(|\mu|)$. Since

$$|\int_K f d\mu| = |\int_K (f - p_k) d\mu| \leq \int_K |f - p_k| d|\mu| \overset{(k\to\infty)}{\longrightarrow} 0$$

and since $f \in C(K)$ was arbitrary, we obtain the contradiction that $\mu = 0$.

□

Since spherical isometries are subnormal, the existence of a spherical isometry as described in Theorem 3.1 is in fact equivalent to the existence of a polynomially convex compact set $K \subset \partial\mathbb{B}$ with $P(K) \neq C(K)$. Indeed, if $S \in L(H)^n$ is subnormal with minimal normal extension $N \in L(K)^n$, then S possesses the contractive functional calculus

$$A(\sigma) \to L(H), \quad f \mapsto f(N)|H,$$

where $\sigma = \sigma(S)$ and $A(\sigma)$ is the closure of $\mathcal{O}(\sigma)|\sigma$ in $C(\sigma)$. Hence, if $A(\sigma) = C(\sigma)$, then S is normal. Therefore, for T as in Theorem 3.1, the set $K = \sigma(T)^\wedge$ is a polynomially convex compact subset of $\partial\mathbb{B}$ with $P(K) \neq C(K)$.
Our next aim is to study more closely the properties of spherical isometries of the above type. The following observation is certainly well known.

Lemma 3.2 *Let $K \subset \partial\mathbb{B}$ be a polynomially convex compact set, and let $\mu \in M^+(K)$ be a measure without atoms. Then the dual algebra $P^\infty(\mu)$ has no w^*–continuous characters.*

Proof. Assume that there is a w^*–continuous character $\varkappa : P^\infty(\mu) \to \mathbb{C}$. Then there is a point $\lambda \in K$ such that $\varkappa(f) = f(\lambda)$ for all $f \in P(K)$. Since λ is a peak point for $A(\mathbb{B})$, there is a function $h \in A(\mathbb{B})$ with $h(\lambda) = 1$ and $|h(z)| < 1$ for all $z \in \overline{\mathbb{B}} \setminus \{\lambda\}$. Because of

$$\int_K \varphi h^k d\mu \overset{(k\to\infty)}{\longrightarrow} \varphi(\lambda)\mu(\{\lambda\}) = 0 \quad (\varphi \in L^1(\mu))$$

it follows that (h^k) induces a w^*–zero sequence in $P^\infty(\mu)$. But then

$$1 = \varkappa(h^k|K) \stackrel{(k\to\infty)}{\longrightarrow} 0.$$

This contradiction shows that the assumption was wrong. □

The last lemma can be used to modify the example given in Theorem 3.1.

Corollary 3.3 *Let $n \geq 3$. Then there is a Henkin measure $\mu \in M^+(\partial\mathbb{B})$ such that*

(i) $S = M_z \in L(P^2(\mu))^n$ is a pure spherical isometry with $\sigma(S)^\wedge \subset \partial\mathbb{B}$,

(ii) $\mathfrak{A}_s \cong P^\infty(\mu)$ has no w^*–continuous characters,

(iii) $P^2(\mu)$ has no bounded point evaluations.

Proof. Let $T \in L(H)^n$ be a non–normal spherical isometry such that the polynomially convex hull K of $\sigma(T)$ is contained in $\partial\mathbb{B}$. Denote by $T_0 \in L(H_0)^n$ ($H_0 \subset H$) its completely non–unitary part and fix $0 \neq h_0 \in H_0$ arbitrary. Then

$$S = T|\bigvee\{T^k h_0;\ k \in \mathbb{N}^n\}$$

is a pure cyclic spherical isometry with $\sigma(S) \subset K$. Since S is cyclic, there is a positive measure μ on K such that $S = M_z \in L(P^2(\mu))^n$ up to unitary equivalence. The measure μ is a Henkin measure because S is pure (Corollary 1.7 in [E1]). In particular, μ has no atoms and by Lemma 3.2 the dual algebra $\mathfrak{A}_s \cong P^\infty(\mu)$ has no w^*–continuous characters. Assume that λ is a bounded point evaluation for $P^2(\mu)$. Then λ belongs to the point spectrum of S^*. In particular, $\lambda \in \partial\mathbb{B}$. Choose a function $h \in A(\mathbb{B})$ with $h(\lambda) = 1 > |h(z)|$ for all $z \in \overline{\mathbb{B}} \setminus \{\lambda\}$. By the dominated convergence theorem, (h^k) converges to zero in $P^2(\mu)$. The contradiction $1 = \lim_{k\to\infty} h^k(\lambda) = 0$ shows that bounded point evaluations cannot exist for $P^2(\mu)$.

□

By the Berger–Shaw theorem each hyponormal operator which possesses a finite rationally cyclic set is essentially normal with trace–class self–commutator. It is well known that no straightforward generalization of this result to the multivariable case is possible. For instance, the commuting tuple $M_z = (M_{z_1}, \ldots, M_{z_n}) \in L(H^2(\mathbb{D}^n))^n$ consisting of the multiplication operators with the coordinate functions on the Hardy space over the unit polydisc is a cyclic subnormal tuple of isometries which is not essentially normal. In the positive direction Douglas and Yan [DY] proved that at least, for a cyclic subnormal

tuple $S = (S_1, \dots, S_n)$ with Taylor spectrum contained in an algebraic curve, all self–commutators $[S_i^*, S_j]$ $(1 \leq i, j \leq n)$ are trace–class operators.
Below we give an example of a pure cyclic spherical isometry $S = (S_1, S_2)$ such that S_2 is self–adjoint, but the pair S is not essentially normal.
Write $I = [0, 1]$ for the unit interval and define $H = H^2(\mathbb{D}) \otimes L^2(I)$. The tuple $S = (S_1, S_2)$ in $L(H)^2$ given by

$$S_1 = M_z \otimes M_\tau, \quad S_2 = I \otimes M_{\sqrt{1-\tau^2}},$$

where τ denotes the independent variable in I, is obviously a spherical isometry which is not essentially normal. A Stone–Weierstraß argument can be used to show that $1 \otimes 1$ is a cyclic vector for S. Alternatively, one can prove directly that S is unitarily equivalent to the multiplication tuple $M_z = (M_{z_1}, M_{z_2}) \in L(P^2(\mu))^2$, where $\mu = \lambda^\varphi$ is the image measure of the planar Lebesgue measure λ restricted to the closed unit disc $\overline{\mathbb{D}}$ under the topological embedding

$$\varphi : \overline{\mathbb{D}} \to \partial\mathbb{B}_2, \quad \varphi(z) = (z, \sqrt{1 - |z|^2}).$$

Since S is a polynomial in the system $T = (M_z \otimes 1,\ 1 \otimes M_\tau, 1 \otimes M_{\sqrt{1-\tau^2}})$, standard results on the spectra of tensor product systems can be used to compute the spectrum and essential spectrum of S

$$\sigma(S) = p(\sigma(T)) = p(\sigma_e(T)) = \sigma_e(S) = \{(rz, \sqrt{1 - r^2});\ z \in \overline{\mathbb{D}} \text{ and } r \in I\},$$

where $p(z_1, z_2, z_3) = (z_1 \cdot z_2, z_3)$.
Since S_1 is a pure subnormal operator, the pair $S = M_z \in L(P^2(\mu))^2$ is a pure spherical isometry. Because of the self–adjoint component S_2, there can be no bounded point evaluations for $P^2(\mu)$ and the dual algebra $\mathfrak{A}_S \cong P^\infty(\mu)$ generated by S cannot possess any w^*–continuous characters. To prove these assertions it suffices to observe that S_2 has empty point spectrum and that $L^\infty(I)$ can be canonically embedded in the algebra $\mathfrak{A}_S \cong P^\infty(\mu)$. As before, since S is a pure spherical isometry, the measure μ is a Henkin measure.

Theorem 3.4 *There is a Henkin measure $\mu \in M^+(\partial\mathbb{B}_2)$ such that*

(i) $S = M_z \in L(P^2(\mu))^2$ *is a pure spherical isometry,*

(ii) S_2 *is self–adjoint, but* S *is not essentially normal,*

(iii) $\mathfrak{A}_S \cong P^\infty(\mu)$ *has no* w^**–continuous characters,*

(iv) $P^2(\mu)$ *has no bounded point evaluations.*

The above examples demonstrate some of the typical phenomena that occur when single–variable isometries are replaced by the much more general concept of a spherical isometry.

4 A decomposition theorem for commuting tuples of operators

In this section we discuss an elementary decomposition and classification of tuples of commuting operators. The decomposition resembles the Jacobi matrix decomposition of a self-adjoint operator A with a cyclic vector x, with respect to the increasing sequence of subspaces generated by succesive powers of A applied to x. In particular our matrix decomposition will apply to spherical isometries and prove Theorem 2.4 above.
Let $S = (S_1, S_2, \dots, S_n) \in L(H)^n$ be a commuting tuple of bounded linear operators on a separable Hilbert space H.

Lemma 4.1 *For any operator $A \in L(H)$, the space*

$$M = \bigcap_{\alpha \in \mathbb{N}^n} \ker(AS^\alpha - S^\alpha A),$$

is the largest invariant subspace for S with the property that

$$AS_j x = S_j A x, \qquad x \in M, 1 \leq j \leq n. \tag{4.1}$$

Proof. For $x \in M$ and $1 \leq j \leq n$ we have

$$AS^\alpha(S_j x) = S^{\alpha+e_j} A x = S^\alpha A(S_j x), \quad \alpha \in \mathbb{N}^n.$$

Conversely, if L is an invariant subspace for S such that relation (4.1) holds for every vector $x \in L$, then an elementary induction shows that $AS^\alpha x = S^\alpha A x$ for all $x \in L$ and $\alpha \in \mathbb{N}^n$.

□

Corollary 4.2 *The space*

$$H_0 = \bigcap_{\beta \in \mathbb{N}^n} \bigcap_{\alpha \in \mathbb{N}^n} \ker(S^{*\beta} S^\alpha - S^\alpha S^{*\beta}),$$

is the largest reducing subspace for S such that $S|_{H_0}$ is normal.

Proof By applying Lemma 4.1 twice we find that H_0 is a reducing subspace for S. Obviously, H_0 is the largest reducing subspace for S such that $S|_{H_0}$ is normal.

□

Lemma 4.3 *For $k \in \mathbb{N}$, the spaces*

$$M_k = \bigcap_{0 \le |\beta| \le k} \bigcap_{\alpha \in \mathbb{N}^n} \ker(S^{*\beta} S^{\alpha} - S^{\alpha} S^{*\beta}),$$

form a decreasing sequence of invariant subspaces for S such that

$$S_j^* M_{k+1} \subset M_k, \qquad k \ge 0, 1 \le j \le n. \tag{4.2}$$

Proof. By Lemma 4.1 it is clear that the spaces M_k are invariant for S. Let $x \in M_{k+1}$ and $j \in \{1, \ldots, n\}$ be given. Then

$$S^{*\beta} S^{\alpha}(S_j^* x) = S^{*(\beta+e_j)} S^{\alpha} x = S^{\alpha} S^{*\beta}(S_j^* x),$$

for all multiindices $\alpha, \beta \in \mathbb{N}^n$ satisfying $|\beta| \le k$.

□

The spaces M_k can be used to construct a canonical decomposition of the space H with respect to the tuple S. Namely, the subspaces $H_p = M_{p-1} \ominus M_p$, $p \ge 1$, satisfy the following relations.

Theorem 4.4 *Let S be a commutative n-tuple of bounded operators on H. There exists a sequence (possibly finite) of pairwise orthogonal subspaces H_p, $p \ge 0$, with the properties:*

(i) $H = \oplus_{p \ge 0} H_p$;

(ii) H_0 *is the largest reducing subspace for* S *such that* $S|_{H_0}$ *is normal;*

(iii) H_1 *is invariant under* S^*;

(iv) $S_j^* H_p \subset H_{p-1} \oplus H_p, \quad p \ge 2, \ 1 \le j \le n$;

(v) $S_j H_p \subset H_p \oplus H_{p+1}, \quad p \ge 1, \ 1 \le j \le n$;

(vi) $H_p \oplus H_{p+1} = (H_p + \sum_{j=1}^n S_j H_p)^-, \quad p \ge 1$;

(vii) $\dim H_{p+1} \le n \dim H_p, \quad p \ge 1$.

□

The standard proof of this result is left to the reader.
From now on we assume that the $n-$tuple S is pure, that is, $H_0 = 0$, and hence $H = \oplus_{p=1}^{\infty} H_p$. For a fixed component $j, 1 \le j \le n$, condition (v) and the decomposition $H = \oplus_{p=1}^{\infty} H_p$ above imply the following matricial structure of S_j:

$$S_j = \begin{pmatrix} D_1^j & 0 & 0 & 0 & \dots \\ C_1^j & D_2^j & 0 & 0 & \dots \\ 0 & C_2^j & D_3^j & 0 & \dots \\ 0 & 0 & C_3^j & D_4^j & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.3}$$

Using condition (vi) in Theorem 4.4 one easily obtains that the operators

$$C_k : H_k^n \longrightarrow H_{k+1}, \quad (x_j) \mapsto \sum_{j=1}^{n} C_k^j x_j,$$

have dense range for $k \ge 1$.
By definition,

$$H_1^{\perp} = M_1 = \bigcap_{j=1}^{n} \bigcap_{\alpha \in \mathbb{N}^n} \ker(S_j^* S^{\alpha} - S^{\alpha} S_j^*) \subset \bigcap_{j,k=1}^{n} \ker[S_j^*, S_k].$$

By taking orthogonal complements this yields

$$\vee_{j,k=1}^{n} \mathrm{Im}[\mathrm{S}_j^*, \mathrm{S}_k] \subset \mathrm{H}_1.$$

This shows that the space H_1 is reducing for all self-commutators $[S_j^*, S_k]$, $1 \le j, k \le n$. Consequently, the matrix of $[S_j^*, S_k]$ with respect to the orthogonal decomposition in Theorem 4.4 contains only one non-zero term, carried by the space H_1.
If one uses the above matrix representation for S_j to compute the self-commutators $[S_j^*, S_k]$ one obtains, for all $1 \le j, k \le n$, the relations

$$[D_1^{j*}, D_1^k] + C_1^{j*} C_1^k = [S_j^*, S_k]|_{H_1}, \tag{4.4}$$

$$[D_p^{j*}, D_p^k] + C_p^{j*} C_p^k = C_{p-1}^k C_{p-1}^{j*}, \quad p \ge 2, \tag{4.5}$$

$$C_p^{j*} D_{p+1}^k = D_p^k C_p^{j*}, \quad p \ge 1. \tag{4.6}$$

Our aim is to prove that the operators

$$D_1^j = (S_j^*|_{H_1})^*, \qquad [S_j^*, S_k]|_{H_1}, \quad 1 \le j, k \le n,$$

form a complete set of unitary invariants for S. First we note that some of the matricial structure of S can be "straightened" into a canonical form.

Proposition 4.5 *Let $S \in L(H)^n$ be a pure commuting n-tuple. Then there are a sequence $(J_p)_{p=1}^{\infty}$ of Hilbert spaces and bounded operators $d_p^j : J_p \longrightarrow J_p$, $c_p : J_p^n \longrightarrow J_p^n$ satisfying*

$$J_{p+1} \subset J_p^n, \quad J_p^n \ominus J_{p+1} = \ker c_p, \quad c_p \geq 0$$

such that S is unitarily equivalent to the tuple $T = (T_1, T_2, \ldots, T_n)$ acting on $J = \oplus_{p=1}^{\infty} J_p$ with components

$$T_j = \begin{pmatrix} d_1^j & 0 & 0 & 0 & \ldots \\ c_1^j & d_2^j & 0 & 0 & \ldots \\ 0 & c_2^j & d_3^j & 0 & \ldots \\ 0 & 0 & c_3^j & d_4^j & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad 1 \leq j \leq n.$$

Here the coefficients $c_p^j : J_p \longrightarrow J_{p+1}$ are determined by the operators c_p via the rule

$$c_p(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} c_p^j x_j.$$

Furthermore, one can achieve that

$$M_p(T) = J_{p+1} \oplus J_{p+2} \oplus \ldots, \quad p \geq 1.$$

Proof. Let $H = \oplus_{p=1}^{\infty} H_p$ be the intrinsic decomposition obtained in Theorem 4.4. We keep the notations explained above. Define $J_1 = H_1$, $W_1 = 1_{J_1}$, and $J_2 = J_1^n \ominus \ker C_1$. Since

$$J_2 \xrightarrow{C_1} H_2$$

is injective with dense range, there exists a unitary operator

$$W_2 : J_2 \longrightarrow H_2.$$

Define $\tilde{C}_1 = W_2^* C_1 \in L(J_1^n, J_2)$ and $\tilde{D}_2^j = W_2^* D_2^j W_2 \in L(J_2)$.
Since the range of the composition

$$J_2^n \xrightarrow{\oplus W_2} H_2^n \xrightarrow{C_2} H_3,$$

is dense, there exists a unitary operator $W_3 : J_3 \longrightarrow H_3$, where $J_3 = J_2^n \ominus \ker(C_2 \circ (\oplus W_2))$. Define as before $\tilde{C}_2$ by $W_3 \circ \tilde{C}_2 = C_2 \circ (\oplus W_2)$, and set $\tilde{D}_3^j = W_3^* D_3^j W_3$, $1 \leq j \leq n$. Continuing in this way one obtains a sequence of spaces J_p with $J_{p+1} \subset J_p^n$, $p \geq 1$ together with operators $\tilde{C}_p : J_p^n \longrightarrow J_{p+1}$, $\tilde{D}_p^j : J_p \longrightarrow J_p$, $W_p : J_p \longrightarrow H_p$, $p \geq 1, 1 \leq j \leq n$, such that, for $p \geq 1$ and $j = 1, \ldots, n$ (with $\tilde{D}_1^j = D_1^j$),

(i) $W_{p+1}\tilde{C}_p = C_p(\oplus W_p)$;

(ii) each W_p is unitary;

(iii) $W_p\tilde{D}_p^j = D_p^j W_p$;

(iv) $\ker \tilde{C}_p = J_p^n \ominus J_{p+1}$;

(v) $\tilde{C}_p$ has dense range.

Note that in this way we obtain a unitary operator

$$W = \oplus W_p : J = \oplus_{p=1}^{\infty} J_p \longrightarrow H = \oplus_{p=1}^{\infty} H_p,$$

which carries the matricial decomposition of each entry of S into a matrix as in the statement, except for one condition: the maps $\tilde{C}_p : J_p^n \longrightarrow J_p^n$ need not be positive. To achieve this last requirement, we rotate again the spaces J_p with the unitaries coming from the polar decomposition of the operators $\tilde{C}_p$ and finally obtain the positive operators c_p in the statement.

□

We shall say that a pure commuting n-tuple T is in *standard form* if the underlying Hilbert space has a decomposition $H = \oplus_{p=1}^{\infty} J_p$ with respect to which T admits a block-matrix representation satisfying all conditions in Proposition 4.5.
The advantage of working with tuples in standard form is illustrated by the natural proof of the next theorem.

Theorem 4.6 *Let $T = (T_1, T_2, \dots, T_n) \in L(H)^n$ and $\tilde{T} = (\tilde{T}_1, \tilde{T}_2, \dots, \tilde{T}_n) \in L(\tilde{H})^n$ be two commuting n-tuples in standard form, relative to the decompositions*

$$H = \oplus_{p=1}^{\infty} J_p, \qquad \tilde{H} = \oplus_{p=1}^{\infty} \tilde{J}_p.$$

Suppose that there exists a unitary operator $U : J_1 \longrightarrow \tilde{J}_1$ such that

$$U(T_j^*|_{J_1})^* = (\tilde{T}_j^*|_{\tilde{J}_1})^* U, \quad U[T_j^*, T_k] = [\tilde{T}_j^*, \tilde{T}_k]U, \quad on \quad J_1,$$

for all $1 \le j, k \le n$.
Then T and $\tilde{T}$ are unitarily equivalent.

Proof. Let T be given by the operators

$$d_p^j : J_p \longrightarrow J_p, \quad c_p : J_p^n \longrightarrow J_p^n, \quad c_p^j : J_p \longrightarrow J_{p+1},$$

and similarly for $\tilde{T}$. Recall that $d_1^j = (T_j^*|_{J_1})^*$. Hence by equation (4.4) and the hypotheses

$$Uc_1^{j*}c_1^k = \tilde{c}_1^{j*}\tilde{c}_1^k U, \quad 1 \leq j,k \leq n.$$

Let $\pi_p : J_p^n \longrightarrow J_{p+1}$ be the orthogonal projection, and let $i_j : J_p \longrightarrow J_p^n$, $i_j(x) = (x\delta_j^i)_{i=1}^n$, be the canonical inclusion.
Since $c_p^j = \pi_p c_p i_j$, we obtain the relations

$$c_p^{j*}c_p^k = i_j^*(c_p\pi_p^*\pi_p c_p)i_k = i_j^* c_p^2 i_k.$$

Therefore $U^{(n)}c_1^2 = \tilde{c}_1^2 U^{(n)}$, and because $c_1, \tilde{c}_1$ are positive elements, we infer that

$$U^{(n)}c_1 = \tilde{c}_1 U^{(n)}.$$

Now, $J_2 = (\ker a_1)^\perp = (\operatorname{Im} c_1)^-$, so that $U^{(n)}J_2 = \tilde{J}_2$. Note that, due to the positivity of c_1, we have $J_1^n = (\operatorname{Im} c_1)^- \oplus \ker c_1$, and that this decomposition is carried over by the unitary map $U^{(n)}$.
Let us write $\alpha_p = (c_p^1, c_p^2, \ldots, c_p^n) : J_p^n \longrightarrow J_{p+1}$ for c_p considered as a map with range space J_{p+1}. Similarly define $\tilde{\alpha}_p$.
According to relations (4.6) we obtain $\alpha_p^* d_{p+1}^k = (\oplus d_p^k)\alpha_p^*$, whence

$$\alpha_p\alpha_p^* d_{p+1}^k = \alpha_p(\oplus d_p^k)\alpha_p^*, \quad p \geq 1, 1 \leq k \leq n.$$

Set $U_1 = U$ and denote by $U_2 : J_2 \longrightarrow \tilde{J}_2$ the unitary map induced by $U^{(n)}$. Since $U_2\alpha_1 = \tilde{\alpha}_1 U_2$, we obtain

$$\tilde{\alpha}_1\tilde{\alpha}_1{}^* U_2 d_2^k = \tilde{\alpha}_1 U_1^{(n)}(\oplus d_1^k)\alpha_1^* = \tilde{\alpha}_1(\oplus d_1^k)U_1^{(n)}\alpha_1^* =$$

$$\tilde{\alpha}_1(\oplus d_1^k)\tilde{\alpha}_1^* U_2 = \tilde{\alpha}_1\tilde{\alpha}_1{}^*\tilde{d}_2^k U_2.$$

But $\ker \tilde{\alpha}_1\tilde{\alpha}_1^* = \ker \tilde{\alpha}_1^* = 0$, hence:

$$U_2 d_2^k = \tilde{d}_2^k U_2, \quad 1 \leq k \leq n. \tag{4.7}$$

Using the observation that

$$U_1^{(n)} c_1 i_k i_j^* c_1 = \tilde{c}_1 U_1^{(n)}(i_k i_j^*)c_1 = \tilde{c}_1(\tilde{i}_k\tilde{i}_j^*)U_1^{(n)}c_1 = \tilde{c}_1(\tilde{i}_k\tilde{i}_j^*)\tilde{c}_1 U_1^{(n)},$$

and the commutation relation $U_2\pi_1 = \tilde{\pi}_1 U_1^{(n)}$, we obtain that

$$U_2 c_1^k c_1^{j*} = U_2(\pi_1 c_1 i_k)(i_j^* c_1 \pi_1^*) = \tilde{\pi}_1\tilde{c}_1\tilde{i}_k\tilde{i}_j^*\tilde{c}_1 U_1^{(n)}\pi_1^* = \tilde{c}_1^k\tilde{c}_1^{j*}U_2.$$

Similarly one proves that

$$U_2 c_1^j = U_2\pi_1 c_1 i_j = \tilde{\pi}_1 U_1^{(n)} c_1 i_j = \tilde{\pi}_1\tilde{c}_1 U_1^{(n)} i_j = \tilde{\pi}_1\tilde{c}_1\tilde{i}_j U_1 = \tilde{c}_1^j U_1.$$

In conclusion, by equations (4.5) and (4.7) we find that

$$U_2 c_2^{j*} c_2^k = \tilde{c}_2^{j*} \tilde{c}_2^k U_2, \quad 1 \le j,k \le n.$$

Continuing in this way one obtains a sequence of unitary operators $U_p : J_p \longrightarrow \tilde{J}_p$ which intertwine the matrix elements d_p^j and c_p^j with the corresponding coefficients in the matrix of $\tilde{T}_j$. . Thus the unitary operator $U = \oplus_{p=1}^{\infty} U_p$ satisfies

$$UT_j = \tilde{T}_j U, \quad 1 \le j \le n,$$

and the proof is complete.

□

Because every pure n-tuple is isomorphic to one in standard form, we have already proved the following result.

Corollary 4.7 *Let $S = (S_1, S_2, \dots, S_n) \in L(H)^n$ and $\tilde{S} = (\tilde{S}_1, \tilde{S}_2, \dots, \tilde{S}_n) \in L(\tilde{H})^n$ be two pure commuting n-tuples.*
Suppose that there exists a unitary operator $U : H_1(S) \longrightarrow H_1(\tilde{S})$ such that

$$U(S_j^*|_{H_1(S)})^* = (\tilde{S}_j^*|_{H_1(\tilde{S})})^* U, \quad U[S_j^*, S_k] = [\tilde{S}_j^*, \tilde{S}_k]U, \quad \text{on} \quad H_1(S),$$

for all $1 \le j,k \le n$.
Then S and $\tilde{S}$ are unitarily equivalent.

□

Let us turn our attention now to the special case where $S \in L(H)^n$ is a subnormal n-tuple. Then the operator

$$D = \sum_{j=1}^{n} [S_j^*, S_j]$$

is positive. The closure of its range $\mathcal{D} = \mathrm{Im}(D)^-$ is called the *defect space* of S. We have

$$\mathcal{D}^{\perp} = \ker D = \bigcap_{j=1}^{n} \ker[S_j^*, S_j].$$

Let $N = (N_1, \dots, N_n) \in L(K)^n$ be the minimal normal extension of S. Each component N_j has a matrix representation

$$N_j = \begin{pmatrix} S_j & U_j \\ 0 & V_j \end{pmatrix}$$

relative to the orthogonal decomposition $K = H \oplus H^\perp$. Since N_j^* commutes with N_k, we obtain the relations

$$[S_j^*, S_k] = U_k U_j^*, \quad S_j^* U_k = U_k V_j^*, \quad 1 \le j, k \le n.$$

Thus

$$D = \sum_{j=1}^{n} U_j U_j^*,$$

and

$$\ker D = \bigcap_{j=1}^{n} \ker U_j U_j^* = \bigcap_{j=1}^{n} \ker U_j^*, \quad \mathcal{D} = \vee_{j=1}^{n} \mathrm{Im}\, U_j.$$

Because of $U_j^* S_k = V_k U_j^*$ it is clear that $\ker U_j^* \in \mathrm{Lat}(S)$. Therefore

$$\ker D \in \mathrm{Lat}(S) \quad \text{and} \quad \mathcal{D} = (\ker D)^\perp \in \mathrm{Lat}(S^*).$$

For $k \in \mathbb{N}$, consider the space

$$\mathcal{D}_k = \vee \{p(S)\mathcal{D};\ \ p \in \mathbb{C}[z_1, z_2, \dots, z_n] \text{ with } \deg(p) \le k\}.$$

Obviously,

$$\mathcal{D}_k = (\sum_{|\beta| \le k} S^\beta \mathrm{Im} D)^-, \tag{4.8}$$

and therefore a short computations shows that

$$\mathcal{D}_k^\perp = \{x \in H;\ \ [S_j^*, S_j] S^{*\beta} x = 0 \text{ for } |\beta| \le k \text{ and } 1 \le j \le n\}.$$

An elementary induction, based on (4.8) and the inclusions $\mathrm{Im}[S_j^*, S_i] \subset \mathcal{D}$, allows one to deduce that all spaces $\mathcal{D}_k$ are invariant for S^*.

Coming back to the notation introduced at the beginning of this section we note the following identity

$$M_{k+1} = \mathcal{D}_k^\perp, \quad k \ge 0. \tag{4.9}$$

Indeed, fix a vector $x \in M_{k+1}$. Then, according to Lemma 4.3, $S_j^* x \in M_k$ and

$$S_j^* S_j S^{*\beta} x = S_j^* S^{*\beta} S_j x = S^{*\beta} S_j (S_j^* x) = S_j S_j^* S^{*\beta} x,$$

for all $|\beta| \le k$ and $1 \le j \le n$. Thus $M_{k+1} \subset \mathcal{D}_k^\perp$.

To prove the converse inclusion we shall use induction on k. For $k = 0$, we have

$$\mathcal{D}_0^\perp = \bigcap_{j=1}^{n} \ker[S_j^*, S_j] \in \mathrm{Lat}(S).$$

Hence, an n-fold application of Lemma 4.1 yields that

$$\mathcal{D}_0^\perp \subset \bigcap_{j=1}^{n} \bigcap_{\alpha \in \mathbb{N}^n} \ker[S_j^*, S^\alpha] = M_1.$$

Suppose that $k \geq 1$ and that $\mathcal{D}_l^\perp \subset M_{l+1},\ 0 \leq l \leq k-1$. This in particular implies $\mathcal{D}_k^\perp \subset \mathcal{D}_{k-1}^\perp \subset M_k$. Fix a vector $x \in \mathcal{D}_k^\perp$. Because of $S_j \mathcal{D}_{k-1} \subset \mathcal{D}_k$, we find $S_j^* x \in S_j^* \mathcal{D}_k^\perp \subset \mathcal{D}_{k-1}^\perp \subset M_k$ for $1 \leq j \leq n$. To see that $x \in M_{k+1}$ it is sufficient to remark that, for any β with $|\beta| = k$ and each index $j = 1, \ldots, n$, we have

$$S^{*(\beta+e_j)} S^\alpha x = S^{*\beta} S^\alpha (S_j^* x) = S^\alpha S^{*(\beta+e_j)} x$$

for all $\alpha \in \mathbb{N}^n$.

As remarked before, for a subnormal tuple, the unitary invariants appearing in Corollary 4.7 are not free. Even for a single subnormal operator the relations among these invariants are rather complicated, see for instance [Pu2].

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