Practice Problems - Solutions Math 34A

These problems were written to be doable without a calculator.

1. Given that $\log(7) = 0.8451$ and $\log(2) = 0.3010$, calculate the following:
(a) log(28)
(b) log(0.0049)
(c) antilog(3.3010)
(d) antilog(-1.1549)

Solution:
(a) $\log(28) = \log(7 \cdot 2 \cdot 2) = \log(7) + \log(2) + \log(2) = 0.8451 + 0.3010 + 0.3010 = \boxed{1.4471}$
(b) $\log(0.0049) = \log(49 \cdot 10^{-4}) = \log(49) + \log(10^{-4}) = \log(7^2) - 4 = 2\log(7) - 4 = 2(.8451) - 4 = \boxed{-2.3098}$
(c) $\text{antilog}(3.3010) = 10^{3.3010} = 10^{3+0.3010} = 10^3 \cdot 10^{0.3010} = 1000 \cdot 2 = \boxed{2000}$
(d) $\text{antilog}(-1.1549) = 10^{-1.1549} = 10^{-2+0.8451} = 10^{-2} \cdot 10^{0.8451} = 0.01 \cdot 7 = \boxed{0.07}$

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2. Suppose a colony of bacteria is growing exponentially. Currently the colony is 6 times as large as it was 5 days ago. What is the doubling time (you may leave logs in your answer)?

Solution:
Recall that the doubling formula is given by $A(t) = A_0 \cdot 2^{\frac{t}{k}}$. The fact that after 5 days the colony is 6 times as large implies that when $t = 5$, $A(5) = 6A_0$.
So $6A_0 = A_0 \cdot 2^{\frac{5}{k}}$. We then want to solve for k:

$$\begin{aligned} 6A_0 &= A_0 \cdot 2^{\frac{5}{k}} \\ \Rightarrow 6 &= 2^{\frac{5}{k}} \\ \Rightarrow \log(6) &= \log(2^{\frac{5}{k}}) \\ \Rightarrow \log(6) &= \frac{5}{k}\log(2) \\ \Rightarrow k &= \boxed{\frac{5\log(2)}{\log(6)}} \end{aligned}$$

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3. Solve for t:
$A \times B^t = \frac{C}{D^{t-2}}$

Solution:

$$\begin{aligned}
A \times B^t &= \frac{C}{D^{t-2}} \\
\Rightarrow A \times B^t &= C \times D^{-(t-2)} \\
\Rightarrow A \times B^t &= C \times D^{-t+2} \\
\Rightarrow \log(A \times B^t) &= \log(C \times D^{-t+2}) \\
\Rightarrow \log(A) + \log(B^t) &= \log(C) + \log(D^{-t+2}) \\
\Rightarrow \log(A) + t\log(B) &= \log(C) + (-t+2)\log(D) \\
\Rightarrow \log(A) + t\log(B) &= \log(C) + -t\log(D) + 2\log(D) \\
\Rightarrow t\log(B) + t\log(D) &= \log(C) + 2\log(D) - \log(A) \\
\Rightarrow t(\log(B) + \log(D)) &= \log(C) + 2\log(D) - \log(A) \\
\Rightarrow t &= \boxed{\frac{\log(C) + 2\log(D) - \log(A)}{\log(B) + \log(D)}}
\end{aligned}$$

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4. Solve for x:
$2log(x+2) = log(x+2) + 1$
Be careful that your solutions for x don't make any of the above logs undefined!

Solution:

$$\begin{aligned}
2\log(x+2) &= \log(x+2) + 1 \\
\Rightarrow \log((x+2)^2) &= \log(x+2) + 1 \\
\Rightarrow 10^{\log((x+2)^2)} &= 10^{\log(x+2)+1} \\
\Rightarrow (x+2)^2 &= 10^{\log(x+2)} \times 10^1 \\
\Rightarrow x^2 + 4x + 4 &= (x+2) \times 10 \\
\Rightarrow x^2 + 4x + 4 &= 10x + 20 \\
\Rightarrow x^2 - 6x - 16 &= 0 \\
\Rightarrow (x-8)(x+2) &= 0
\end{aligned}$$

Thus $x = -2, 8$. However, $x = -2$ would mean the above logs are undefined, since $log(x+2) = \log(0)$. Thus the only legal solution is $\boxed{x = 8}$.

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5. Andy invests \$100 in an account that gains interest at a rate of 3% a year. Bob invests \$50 in an account that gains interest at a rate of 2% every *half* a year. How many years would it take until Bob's account has the same amount of money as Andy?

Solution:

Let $A(t)$ represent the amount in Andy's account after t years. Then as Andy starts with \$100 and gains interest at 3% a year, we get that $A(t) = 100 \times (1.03)^t$, since gaining 3% interest is the same as multiplying by 1.03.

Similarly, let $B(t)$ be the amount in Bob's account after t years. Since Bob's money gains interest every half a year, we get that $B(t) = 50 \times (1.02)^{2t}$. One way to think of this is that, for every year that Andy's money gains 3% interest, Bob's money gains 2% interest *twice*.

To find out when Bob and Andy have the same amount of money, we set $A(t) = B(t)$ and solve for t:

$$\begin{aligned}
A(t) &= B(t) \\
\Rightarrow 100(1.03)^t &= 50(1.02)^{2t} \\
\Rightarrow 2(1.03)^t &= (1.02)^{2t} \\
\Rightarrow \log(2(1.03)^t) &= \log(1.02^{2t}) \\
\Rightarrow \log(2) + \log(1.03^t) &= \log(1.02^{2t}) \\
\Rightarrow \log(2) + t\log(1.03) &= 2t\log(1.02) \\
\Rightarrow 2t\log(1.02) - t\log(1.03) &= \log(2) \\
\Rightarrow t(2\log(1.02) - \log(1.03)) &= \log(2) \\
\Rightarrow t &= \boxed{\frac{\log(2)}{2\log(1.02) - \log(1.03)}}
\end{aligned}$$

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6. A certain radioactive isotope decays exponentially with half-life 84 years. If we start with 32 grams of the isotope, how many years will it take until there are 7 grams left? You may leave logs in your answer.

Solution:

Using our half-life equation, we get that $A(t) = 32 \times \frac{1}{2}^{t/84}$. We then want to solve for the t such that $A(t) = 7$. Substituting 7 for $A(t)$ in our equation and solving gives:

$$
\begin{aligned}
7 &= 32 \times \frac{1}{2}^{t/84} \\
\Rightarrow \frac{7}{32} &= \frac{1}{2}^{t/84} \\
\Rightarrow \log(\frac{7}{32}) &= \log(\frac{1}{2}^{t/84}) \\
\Rightarrow \log(\frac{7}{32}) &= \frac{t}{84}\log(\frac{1}{2}) \\
\Rightarrow t &= \boxed{\frac{84\log(\frac{7}{32})}{\log(\frac{1}{2})}}
\end{aligned}
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7. Suppose that a colony of 20 rabbits in 1990 increased to a population of 140 in 2002. Assuming the population of rabbits is increasing exponentially, how many rabbits will there be in 2014?

Solution:

Given how the data is given, it is easier to say that the number of rabbits is multiplied by 7 every 12 years than trying to calculate a doubling time. So this gives us that the number of rabbits $R(t)$, t years after 1990, is $R(t) = 20 \times 7^{t/12}$.

To determine the number of rabbits in 2010, we then set $t = 24$ to get $R(20) = 20 \times 7^{24/12} = 20 \times 7^2 = 20 \times 49 = \boxed{\text{980 rabbits}}$.

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8. Solve for x using natural logs:
$10^{x+4} = 8e^{3x}$

Solution:

$$\begin{aligned}
10^{x+4} &= 8e^{3x} \\
\Rightarrow \ln(10^{x+4}) &= \ln(8e^{3x}) \\
\Rightarrow \ln(10^{x+4}) &= \ln(8) + \ln(e^{3x}) \\
\Rightarrow (x+4)\ln(10) &= \ln(8) + 3x\ln(e) \\
\Rightarrow (x+4)\ln(10) &= \ln(8) + 3x \\
\Rightarrow x\ln(10) + 4\ln(10) &= \ln(8) + 3x \\
\Rightarrow 3x - x\ln(10) &= 4\ln(10) - \ln(8) \\
\Rightarrow x(3 - \ln(10)) &= 4\ln(10) - \ln(8) \\
\Rightarrow x &= \boxed{\frac{4\ln(10) - \ln(8)}{3 - \ln(10)}}
\end{aligned}$$

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9. Find the average rate of change of $f(x) = 5x^2 - 3$ between $x = 2$ and $x = 3$. Now find the average rate of change between $x = 2$ and $x = 2 + h$. What is the instantaneous rate of change at $x = 2$?

Solution:
By definition, the average rate of change of $f(x)$ between $x = 2$ and $x = 3$ is:
$\frac{f(3) - f(2)}{3 - 2} = \frac{(5(3)^2 - 3) - (5(2)^2 - 3)}{1} = (5 \cdot 9 - 3) - (5 \cdot 4 - 3) = 42 - 17 = \boxed{25}$
The average rate of change of $f(x)$ between $x = 2$ and $x = 2 + h$ is:

$$\begin{aligned}
\frac{f(2+h) - f(2)}{(2+h) - (2)} &= \frac{(5(2+h)^2 - 3) - (5(2)^2 - 3)}{h} \\
&= \frac{(5(4 + 4h + h^2) - 3) - (5(4) - 3)}{h} \\
&= \frac{17 + 20h + 5h^2 - 17}{h} \\
&= \frac{20h + 5h^2}{h} \\
&= \boxed{20 + 5h}
\end{aligned}$$

The instantaneous rate of change at $x = 2$ is:
$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0}(20 + 5h) = \boxed{20}$

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10. Using the limit definition of the derivative, find $f'(x)$ when $f(x) = 3x^2 - 2$.

Solution:

The limit definition is $f'(x) = \lim\limits_{x \to \infty} \dfrac{f(x+h) - f(x)}{h}$.

Since $f(x) = 3x^2 - 2$ we then have:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(3(x+h)^2 - 2) - (3(x)^2 - 2)}{h} \\
&= \lim_{h \to 0} \frac{(3(x^2 + 2xh + h^2) - 2) - (3x^2 - 2)}{h} \\
&= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 2 - 3x^2 + 2}{h} \\
&= \lim_{h \to 0} \frac{6xh + 3h^2}{h} \\
&= \lim_{h \to 0} 6x + 3h \\
&= \boxed{6x}
\end{aligned}$$

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11. Suppose an object is thrown off the top of a building and its height in meters after t seconds is given by the function $f(t) = 50 + 5t - t^2$. How fast is the object traveling when it hits the ground? When does the object attain its maximum height?

Solution:

Since $f(t)$ represents the height of the object, to find out the time when the object hits the ground we set $f(t) = 0$ and solve for t:

$$\begin{aligned}
f(t) &= 0 \\
50 + 5t - t^2 &= 0 \\
(10 - t)(5 + t) &= 0
\end{aligned}$$

So $t =$ -5 or 10, so we take $t = 10$ since we want a positive value for time. Then to know how fast the object was traveling when it hits the ground, we substitute $t = 10$ into the derivative. $f'(t) = 5 - 2t$ so $f'(10) = 5 - 2 \cdot 10 = -15$. Hence the object was moving at $\boxed{15\text{m/s}}$ when it hit the ground.

When the object attains its maximum height it has derivative 0, so we find the value of t corresponding to $f'(t) = 0$. $5 - 2t = 0$ implies that $t = \frac{5}{2}$. So the object attains its maximum height at $\boxed{\frac{5}{2}\text{s}}$.

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12. Evaluate the following derivatives:

(a) $\frac{d}{dx}(4e^{x/2})$

(b) $\frac{d}{dx}(\frac{x^2-3}{x})$

(c) $\frac{d}{dx}(\frac{2}{\sqrt[5]{x}})$

(d) $\frac{d}{dx}(\frac{x^2-4}{x+2})$

Solution:

(a) $\frac{d}{dx}(4e^{x/2}) = \frac{1}{2}\cdot 4e^{x/2} = \boxed{2e^{x/2}}$

(b) $\dfrac{d}{dx}\left(\dfrac{x^2-3}{x}\right) = \dfrac{d}{dx}\left(\dfrac{x^2}{x} - \dfrac{3}{x}\right) = \dfrac{d}{dx}\left(x - \dfrac{3}{x}\right) = \dfrac{d}{dx}\left(x - 3x^{-1}\right) = 1 + 3x^{-2} = \boxed{1 + \dfrac{3}{x^2}}$

(c) $\dfrac{d}{dx}\left(\dfrac{2}{\sqrt[5]{x}}\right) = \dfrac{d}{dx}\left(2x^{-1/5}\right) = -\dfrac{1}{5}\cdot 2x^{-6/5} = \boxed{-\dfrac{2}{5}x^{-6/5}}$

(d) $\dfrac{d}{dx}\left(\frac{x^2-4}{x+2}\right) = \dfrac{d}{dx}\left(\dfrac{(x+2)(x-2)}{x+2}\right) = \dfrac{d}{dx}(x-2) = \boxed{1}$

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13. Let $f(x) = \frac{1}{x}$. Find an equation for the tangent line to $f(x)$ at $x = 2$. Then use this tangent line to approximate $\frac{1}{3}$. What is the resulting percent error?

Solution:

The slope of this tangent line will be $f'(2)$. Since $f'(x) = -\frac{1}{x^2}$, we get that $f'(2) = -\frac{1}{4}$. The tangent line will also intersect the graph of $f(x)$ at $x = 2$, so it must pass through the point $(2, f(2)) = (2, \frac{1}{2})$. Using point-slope form, we then get the equation:

$$\begin{aligned} y - \frac{1}{2} &= -\frac{1}{4}(x-2) \\ \Rightarrow y &= -\frac{1}{4}x + \frac{1}{2} + \frac{1}{2} \\ \Rightarrow y &= -\frac{1}{4}x + 1 \end{aligned}$$

So the tangent line to $f(x) = \frac{1}{x}$ at $x = 2$ is $\boxed{y = -\frac{1}{4}x + 1}$.

To approximate $\frac{1}{3}$ with the tangent line that is approximating $f(x) = \frac{1}{x}$, we set $x = 3$ to get $y = -\frac{1}{4}\cdot(3) + 1 = -\frac{3}{4} + 1 = \boxed{\frac{1}{4}}$.

The resulting percent error is then:

$\dfrac{\left|\frac{1}{3} - \frac{1}{4}\right|}{\frac{1}{3}} \times 100\% = \dfrac{\frac{1}{12}}{\frac{1}{3}} \times 100\% = \dfrac{1}{12}\cdot\dfrac{3}{1} \times 100\% = \dfrac{1}{4} \times 100\% = 25\%.$

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14. Let $f(x) = -x^3 + 6x^2 - 12x - 5$. For what values of x is $f(x)$ decreasing? For what values of x is $f(x)$ concave down?

Solution:
To determine where $f(x)$ is decreasing, we need to first find its derivative:
$f'(x) = -3x^2 + 12x - 12$
We then set $f'(x)$ to zero to find critical points.

$$\begin{aligned} -3x^2 - 12x + 12 &= 0 \\ \Rightarrow x^2 - 4x + 4 &= 0 \\ \Rightarrow (x-2)(x-2) &= 0 \end{aligned}$$

So there is a critical point at $x = 2$ and therefore we just need to test the sign of $f'(x)$ in the regions $x < 2$ and $x > 2$. $f'(0) = -12 < 0$ and $f'(3) = -3 < 0$ so $f'(x)$ is negative on both intervals. Therefore $f(x)$ is decreasing on both $\boxed{x < 2 \text{ and } x > 2}$.
To determine where $f(x)$ is concave down, we need to find its second derivative:
$f''(x) = -6x + 12$
Setting $f''(x) = 0$ then gives us $-6x + 12 = 0$ so $x = 2$ is our inflection point. So now we need to test the sign of $f''(x)$ in the regions $x < 2$ and $x > 2$. $f''(0) = 12 > 0$ and $f''(3) = -6 < 0$. So $f''(x)$ is negative only on the region $x > 2$. So $f(x)$ is concave down on $\boxed{x > 2}$.

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15. Suppose $T(p)$ represents the time it takes in minutes to get a sandwich at the Arbor Subway when there are p people in line. What does $T(3) = 5$ mean? What does $T'(3) = 1$ mean? What does $T^{-1}(8) = 6$ mean?

Solution:
$T(3) = 5$ means when there are 3 people in line, it takes 5 minutes to get a sandwich.
$T'(3) = 1$ means when there are 3 people in line, the time it takes to get a sandwich is increasing at a rate of 1 minute per person in line.
$T^{-1}(8) = 6$ means it takes 8 minutes to get a sandwich when there are 6 people in line.

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16. Suppose $f(t)$ represents the amount of liters in a water tank after t days. What does $f'(4) = -2$ mean? What does $f''(4) = .5$ mean? If $f''(t)$ is always equal to .5, at what time will the tank no longer be losing water?

Solution:
$f'(4) = -2$ means that at 4 days, the amount of water in the tank is decreasing at a rate of 2 liters/day.
$f''(4) = .5$ means that at 4 days, the rate of water entering/leaving the tank is increasing at a rate of 0.5 liters/day each day (alternatively, it is increasing at a rate of 0.5 liters/day^2).
If $f''(t) = 0.5$ for all t then after each day $f'(t)$ will increase by 0.5. Thus since $f'(4) = -2$, $f'(5)$ will be -1.5, $f'(6)$ will be -1, and so on and then clearly $f'(8) = 0$ and the tank will no longer be losing water. So the tank will no longer be losing water starting on $\boxed{\text{day 8}}$.

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17. Suppose some kids decide to open a lemonade stand on a street corner. It costs them 20 cents to make a single cup of lemonade. After weeks of selling, they find that they can sell 50 cups when they charge \$1 per cup and that their sales fall by 5 for each 10 cents extra that they charge. What price should the kids sell their lemonade to make the most profit? What is the maximum profit they can make?

Solution:

Let n be the number of cups of lemonade the kids sell and p the price for each cup. Then the Profit (P) function for the kids will be $P = np - 0.2n = n(p - 0.2)$ since np represents the amount of income they make and the $-0.2n$ represents the 20 cents cost for making each cup of lemonade. The question asks for the ideal *price* to sell the lemonade, so we want a function for profit in terms of just p.
The fact that the kids sell 50 cups when they charge \$1 per cup gives us a point (1,50) if we graphed p versus n. The fact that sales fall by 5 for each 10 cents extra they charge gives us a slope of $m = -\frac{5}{.1} = -50$.
Point-slope form then gives us:

$$\begin{aligned} n - 50 &= -50(p - 1) \\ \Rightarrow n - 50 &= -50p + 50 \\ \Rightarrow n &= -50p + 50 + 50 \\ \Rightarrow n &= -50p + 100 \end{aligned}$$

Substituting this formula for n into our equation for profit gives us:

$$\begin{aligned} P(p) &= (-50p + 100)(p - 0.2) \\ &= -50p^2 + 100p + 10p - 20 \\ &= -50p^2 + 110p - 20 \end{aligned}$$

So $P'(p) = -100p + 110$ and solving for critical points yields $-100p + 110 = 0 \Rightarrow p = 1.1$. Thus the price that yields the maximum profit is $\boxed{\$1.10}$.
The maximum profit the kids can make then requires substituting this ideal price back into the function for profit. So $P(1.10) = (-50 \cdot 1.10 + 100)(1.10 - 0.2) = (-55 + 100)(0.9) = (45)(0.9) = 40.50$. Thus the kids' maximum profit is $\boxed{\$40.50}$.

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18. Suppose a farmer wants to make a rectangular field that is bordered by a brick wall on one side and wooden fence on the other three sides. The cost of the brick wall is \$5 per meter and the wooden fence costs \$3 per meter. If the farmer only has \$120 to spend, what is the maximum area this field can be?

Solution:

Let L be the length of the brick side of the rectangular perimeter and W denote the other dimension. Then the cost of the entire perimeter would be L meters of brick wall at \$5 per meter and $(2W + L)$ meters of wooden fence at \$3 per meter, yielding Cost $= 5L + 3(2W + L) = 8L + 6W$.
Then since the farmer only has \$120 to spend, we set $8L + 6W = 120$ as we assume the farmer will need to spend all of his money to maximize the area of his field.
Now we want to maximize the area (A) of the field, where $A = WL$. We would like to get a formula for the area in terms of one variable. We can use the above cost formula to do this. $8L + 6W = 120$ implies $W = \dfrac{120 - 8L}{6}$. Substituting this into the area equation gives us:
$A(L) = \left(\dfrac{120 - 8L}{6}\right)L = \dfrac{120L - 8L^2}{6} = 20L - \dfrac{4}{3}L^2$
$\Rightarrow A'(L) = 20 - \dfrac{8}{3}L$
Setting to 0 and solving then yields $20 - \dfrac{8}{3}L = 0 \Rightarrow L = \dfrac{20 \cdot 3}{8} = \dfrac{15}{2}$. Then the maximum area is then:

$$\begin{aligned}
A\left(\frac{15}{2}\right) &= 20 \cdot \frac{15}{2} - \frac{4}{3}\left(\frac{15}{2}\right)^2 \\
&= 150 - \frac{4}{3}\left(\frac{225}{4}\right) \\
&= 150 - \frac{225}{3} \\
&= 150 - 75 \\
&= \boxed{75 \text{ m}^2}
\end{aligned}$$