Practice Problems - Solutions
Math 34A

These problems were written to be doable without a calculator.

1. Let $f(x) = \frac{2x+a}{ax+3}$. Find the inverse function of $f(x)$.

Solution:

Let $y = f(x)$.
Then $y = \frac{2x+a}{ax+3}$.
Solving for x:

$$\begin{aligned} y(ax+3) &= 2x+a \\ \Rightarrow axy + 3y &= 2x + a \\ \Rightarrow axy - 2x &= a - 3y \\ \Rightarrow x(ay-2) &= a - 3y \\ \Rightarrow x &= \frac{a-3y}{ay-2} \end{aligned}$$

$y = f(x)$ implies $x = f^{-1}(y)$, so $\boxed{f^{-1}(y) = \frac{a-3y}{ay-2}}$.

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2. Find where the line running through $(3, 2)$ and the origin intersects the line running through $(0, 4)$ and $(-1, 2)$.

Solution:

Equation of line running through $(3, 2)$ and the origin $(0, 0)$:
$m = \frac{2-0}{3-0} = \frac{2}{3}$
So using point-slope form with the point $(0, 0)$ gives:
$y - 0 = \frac{2}{3}(x - 0) \Rightarrow y = \frac{2}{3}x$
Equation of line running through $(0, 4)$ and $(-1, 2)$:
$m = \frac{2-4}{-1-0} = \frac{-2}{-1} = 2$
So using point-slope form with the point $(0, 4)$ gives:
$y - 4 = 2(x - 0) \Rightarrow y = 2x + 4$
Find the point of intersection of these two lines:
We want to find the point (x, y) satisfying both $y = \frac{2}{3}x$ and $y = 2x + 4$.
Substituting for y, we get:

$$\begin{aligned} \frac{2}{3}x &= 2x + 4 \\ \Rightarrow -\frac{4}{3}x &= 4 \\ \Rightarrow x &= -3 \end{aligned}$$

We then substitute $x = -3$ back into one of the line equations (I'll use $y = \frac{2}{3}x$):
$y = \frac{2}{3} \cdot -3 = -2$
So the point of intersection is $\boxed{(-3, -2)}$.

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3. At noon, Person A starts walking due north at 3 mph. Person B then leaves from the exact same starting point an hour later and jogs west at 6 mph. At what time are they 15 miles apart?

Solution:

Let t be hours after 12pm. Then the distance traveled by Person A is $3t$ and the distance traveled by Person B is $6(t-1) = 6t - 6$. Since the two people are moving in perpendicular directions, we can use Pythagoras' Theorem to get that the squared distance between the two people is $(3t)^2 + (6t-6)^2$. Based on the question, we then want to solve for t when $(3t)^2 + (6t-6)^2 = 15^2$:

$$\begin{aligned} (3t)^2 + (6t-6)^2 &= 15^2 \\ \Rightarrow 9t^2 + 36t^2 - 72t + 36 &= 225 \\ \Rightarrow 45t^2 - 72t + 36 &= 225 \\ \Rightarrow 5t^2 - 8t + 4 &= 25 \text{ (divided by 9)} \\ \Rightarrow 5t^2 - 8t - 21 &= 0 \\ \Rightarrow (5t+7)(t-3) &= 0 \end{aligned}$$

So $t = 3$ or $-7/5$, but we want a positive value for t so we'll discard the negative solution. Thus the two people will be 15 miles apart at $\boxed{\text{3pm}}$.

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4. Suppose you work at a paint store and a customer wants 3 liters of a particular shade of purple that is 40% blue and 60% red. Unfortunately, the store is out of red and blue paint! However it does have two shades of purple in stock: one purple that is 32% blue and 68% red and the other 52% blue and 48% red. How much of each of these two shades of purple needs to be mixed together to meet the customer's wishes?

Solution:

Let x be the number of liters of the (32% blue/68% red) purple and y be the number of liters of the (52% blue/48% red) purple.

Then the condition that the customer wants a total of 3 liters implies: $x + y = 3$.

The fact that the customer wants a purple that is 40% blue means: $.32x + .52y = .40 \times 3 = 1.2$.

If the resulting purple is 40% blue then it will be automatically be 60% red, so we do not need to write an equation about the red paint.

Multiplying the second equation by 100 gives: $32x + 52y = 120$. Multiplying the first equation by -32 gives: $-32x - 32y = -96$. Adding the two equations together then gives:

$$\begin{aligned} 32x + 52y &= 120 \\ -32x - 32y &= -96 \\ \hline 20y &= 24 \end{aligned}$$

So $y = \frac{24}{20} = 1.2$. Then $x + y = 3$ implies $x = 1.8$. So we need to mix together $\boxed{\text{1.8 liters}}$ of the (32% blue/68% red) purple and $\boxed{\text{1.2 liters}}$ of the (52% blue/48% red) purple.

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5. Suppose there is a circular swimming pool that is 6 meters in diameter and 1.5 meters deep. For some reason we want to fill up this pool using a garden hose, which shoots out water at a rate of 10 liters per minute. How many hours will it take to fill the pool? Note that 1 liter = 1000 cm^3.

Solution:

The volume of the pool is $\pi r^2 h = \pi \cdot 3^2 \cdot 1.5 = 13.5\pi$ m^3.

Then the time it takes for the hose to fill the pool is:

$$\begin{aligned}\frac{13.5\pi \text{ m}^3}{10\text{L/ minute}} &= \frac{13.5\pi \text{ m}^3 \cdot \text{ minutes}}{10 \text{ L}} \times \frac{1 \text{ L}}{1000 \text{ cm}^3} \times \frac{1000000 \text{ cm}^3}{1 \text{ m}^3} \\ &= 1350\pi \text{ minutes} \times \frac{1 \text{ hour}}{60 \text{ minutes}} \\ &= \boxed{22.5\pi \text{ hours}}\end{aligned}$$

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6. Find the following limits:

$\lim_{x\to\infty} \frac{5x^2+3x}{4-2x^2}$.

$\lim_{x\to 2} \frac{x^2+2x-15}{x-3}$.

$\lim_{n\to\infty} x_n$ when $x_1 = 1$, $x_2 = -\frac{1}{2}$, $x_3 = \frac{1}{3}$, $x_4 = -\frac{1}{4}$, ...

Solution:

$$\begin{aligned}\lim_{x\to\infty} \frac{5x^2+3x}{4-2x^2} &= \lim_{x\to\infty} \frac{\frac{5x^2}{x^2} + \frac{3x}{x^2}}{\frac{4}{x^2} - \frac{2x^2}{x^2}} \\ &= \lim_{x\to\infty} \frac{5 + \frac{3}{x}}{\frac{4}{x^2} - 2} \\ &= \frac{5+0}{0-2} \\ &= \boxed{-\frac{5}{2}}\end{aligned}$$

$$\begin{aligned}\lim_{x\to 2} \frac{x^2+2x-15}{x-3} &= \lim_{x\to 2} \frac{(x+5)(x-3)}{x-3} \\ &= \lim_{x\to 2}(x+5) \\ &= 2+5 \\ &= \boxed{7}\end{aligned}$$

$\lim_{n\to\infty} x_n = \boxed{0}$ as x_n gets closer to 0 as n increases.

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7. Suppose that in 1960 there were 10,000 students at UCSB and in 2000 there were 18,000 students at UCSB. Using linear extrapolation, predict how many students there will be at UCSB in 2020. Now suppose that in 2020 the actual student body was 20,000. What would our percent error be?

Solution:

Let x represent the number years after 1900 and y the number of students at UCSB during year x. Then we can interpret the data above as the two data points (60,10000) and (100,18000). The slope of the line through these two points is:

$m = \frac{18000-10000}{100-60} = \frac{8000}{40} = \frac{800}{4} = 200$

Using point-slope form with the point (60,10000) then gives:

$y - 10000 = 200(x - 60) \Rightarrow y - 10000 = 200x - 12000 \Rightarrow y = 200x - 2000$

Then to extrapolate what the student body is in 2020, we set $x = 120$ to get:

$y = 200 \cdot 120 - 2000 = 24000 - 2000 = \boxed{22000 \text{ students}}$.

If the actual student body turns out to be 20,000, then our percent error is:

$$\frac{|22000 - 20000|}{200000} \times 100\% = \frac{2000}{20000} \times 100\% = \frac{1}{10} \times 100\% = \boxed{10\%}$$

8. Combine the following into a single summation:

$$\sum_{n=1}^{100} x_n - \sum_{n=50}^{100} x_n$$

$$\sum_{n=1}^{100} x_n + \sum_{n=1}^{100} y_n$$

Calculate the following:

$$\sum_{n=1}^{100} 2$$

$$\sum_{m=1}^{3} \left(\sum_{n=1}^{2} \left(\frac{m}{n} \right) \right)$$

Solution:

$$\sum_{n=1}^{100} x_n - \sum_{n=50}^{100} x_n = \sum_{n=1}^{49} x_n$$

$$\sum_{n=1}^{100} x_n + \sum_{n=1}^{100} y_n = \sum_{n=1}^{100} (x_n + y_n)$$

$$\sum_{n=1}^{100} 2 = 2 + 2 + \ldots 2 \text{ (100 times)} = 2 \cdot 100 = \boxed{200}$$

$$\begin{aligned} \sum_{m=1}^{3} \left(\sum_{n=1}^{2} \left(\frac{m}{n} \right) \right) &= \sum_{m=1}^{3} \left(\frac{m}{1} + \frac{m}{2} \right) \\ &= \sum_{m=1}^{3} \left(\frac{3m}{2} \right) \\ &= \frac{3}{2} + \frac{6}{2} + \frac{9}{2} \\ &= \frac{18}{2} \\ &= \boxed{9} \end{aligned}$$

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9. Suppose we have two boxes A and B, but the sides of Box A are 3 times the length of the sides of Box B. How much more volume can Box A hold compared to Box B?

Solution:

Suppose the dimensions of Box B are x, y, and z, so Box B has volume xyz. Then the dimensions of Box A are $3x$, $3y$, and $3z$, so its volume is $(3x)(3y)(3z) = 27xyz$. So the volume of Box A is $\boxed{\text{27 times}}$ that of Box B.

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10. Suppose the time it takes before a car's brakes wear out is inversely proportional to both the average number of miles the car travels each day and how heavy the car is. Suppose the brakes of a 3 ton truck driven 30 miles a day need replacing after 6 years. Then if a 2 ton sedan's brakes need replacing after 5 years, on average how many miles was it driven per day?

Solution:

Let T represent the time it takes before a car's brakes wear out.
Let W represent the weight of the car.
Let M represent the average number of miles the car travels each day.
Then $T = \frac{K}{WM}$ where K is a proportionality constant.
The fact that a 3 ton truck driven 30 miles a day needs its brakes replaced after 6 years implies:
$6 = \frac{K}{3 \cdot 30}$, which implies $K = 6 \cdot 3 \cdot 30 = 540$. Using $K = 540$ in the second scenario then gives us:
$5 = \frac{540}{2 \cdot M} \Rightarrow M = \frac{540}{2 \cdot 5} = 54$. So the car was driven an average of $\boxed{\text{54 miles / day}}$.

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11. Given that log(7) = 0.8451 and log(2) = 0.3010, calculate the following:
(a) log(28)
(b) log(0.0049)
(c) antilog(3.3010)
(d) antilog(-1.1549)

Solution:

(a) $\log(28) = \log(7 \cdot 2 \cdot 2) = \log(7) + \log(2) + \log(2) = 0.8451 + 0.3010 + 0.3010 = \boxed{1.4471}$
(b) $\log(0.0049) = \log(49 \cdot 10^{-4}) = \log(49) + \log(10^{-4}) = \log(7^2) - 4 = 2\log(7) - 4 = 2(.8451) - 4 = \boxed{-2.3098}$
(c) $\text{antilog}(3.3010) = 10^{3.3010} = 10^{3+0.3010} = 10^3 \cdot 10^{0.3010} = 1000 \cdot 2 = \boxed{2000}$
(d) $\text{antilog}(-1.1549) = 10^{-1.1549} = 10^{-2+0.8451} = 10^{-2} \cdot 10^{0.8451} = 0.01 \cdot 7 = \boxed{0.07}$

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12. Suppose a colony of bacteria is growing exponentially. Currently the colony is 6 times as large as it was 5 days ago. What is the doubling time (you may leave logs in your answer)?

Solution:

Recall that the doubling formula is given by $A(t) = A_0 \cdot 2^{\frac{t}{k}}$. The fact that after 5 days the colony is 6 times as large implies that when $t = 5$, $A(5) = 6A_0$.
So $6A_0 = A_0 \cdot 2^{\frac{5}{k}}$. We then want to solve for k:

$$\begin{aligned} 6A_0 &= A_0 \cdot 2^{\frac{5}{k}} \\ \Rightarrow 6 &= 2^{\frac{5}{k}} \\ \Rightarrow \log(6) &= \log(2^{\frac{5}{k}}) \\ \Rightarrow \log(6) &= \frac{5}{k}\log(2) \\ \Rightarrow k &= \boxed{\frac{5\log(2)}{\log(6)}} \end{aligned}$$

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13. Solve for t:
$A \times B^t = \frac{C}{D^{t-2}}$

Solution:

$$\begin{aligned}
A \times B^t &= \frac{C}{D^{t-2}} \\
\Rightarrow A \times B^t &= C \times D^{-(t-2)} \\
\Rightarrow A \times B^t &= C \times D^{-t+2} \\
\Rightarrow \log(A \times B^t) &= \log(C \times D^{-t+2}) \\
\Rightarrow \log(A) + \log(B^t) &= \log(C) + \log(D^{-t+2}) \\
\Rightarrow \log(A) + t\log(B) &= \log(C) + (-t+2)\log(D) \\
\Rightarrow \log(A) + t\log(B) &= \log(C) + -t\log(D) + 2\log(D) \\
\Rightarrow t\log(B) + t\log(D) &= \log(C) + 2\log(D) - \log(A) \\
\Rightarrow t(\log(B) + \log(D)) &= \log(C) + 2\log(D) - \log(A) \\
\Rightarrow t &= \boxed{\frac{\log(C) + 2\log(D) - \log(A)}{\log(B) + \log(D)}}
\end{aligned}$$

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14. Find the average rate of change of $f(x) = 5x^2 - 3$ between $x = 2$ and $x = 3$. Now find the average rate of change between $x = 2$ and $x = 2 + h$. What is the instantaneous rate of change at $x = 2$?

Solution:

By definition, the average rate of change of $f(x)$ between $x = 2$ and $x = 3$ is:
$\dfrac{f(3) - f(2)}{3 - 2} = \dfrac{(5(3)^2 - 3) - (5(2)^2 - 3)}{1} = (5 \cdot 9 - 3) - (5 \cdot 4 - 3) = 42 - 17 = \boxed{25}$

The average rate of change of $f(x)$ between $x = 2$ and $x = 2 + h$ is:

$$\begin{aligned}
\frac{f(2+h) - f(2)}{(2+h) - (2)} &= \frac{(5(2+h)^2 - 3) - (5(2)^2 - 3)}{h} \\
&= \frac{(5(4 + 4h + h^2) - 3) - (5(4) - 3)}{h} \\
&= \frac{17 + 20h + 5h^2 - 17}{h} \\
&= \frac{20h + 5h^2}{h} \\
&= \boxed{\text{20+5h}}
\end{aligned}$$

The instantaneous rate of change at $x = 2$ is:
$\lim\limits_{h \to 0} \dfrac{f(2+h) - f(2)}{h} = \lim\limits_{h \to 0}(20 + 5h) = \boxed{20}$

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15. Using the limit definition of the derivative, find $f'(x)$ when $f(x) = 3x^2 - 2$.

Solution:

The limit definition is $f'(x) = \lim_{x\to\infty} \frac{f(x+h)-f(x)}{h}$.

Since $f(x) = 3x^2 - 2$ we then have:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(3(x+h)^2-2)-(3(x)^2-2)}{h} \\
&= \lim_{h\to 0} \frac{(3(x^2+2xh+h^2)-2)-(3x^2-2)}{h} \\
&= \lim_{h\to 0} \frac{3x^2+6xh+3h^2-2-3x^2+2}{h} \\
&= \lim_{h\to 0} \frac{6xh+3h^2}{h} \\
&= \lim_{h\to 0} 6x+3h \\
&= \boxed{\text{6x}}
\end{aligned}
$$

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16. Suppose an object is thrown off the top of a building and its height in meters after t seconds is given by the function $f(t) = 50 + 5t - t^2$. How fast is the object traveling when it hits the ground? When does the object attain its maximum height?

Solution:

Since $f(t)$ represents the height of the object, to find out the time when the object hits the ground we set $f(t) = 0$ and solve for t:

$$
\begin{aligned}
f(t) &= 0 \\
50 + 5t - t^2 &= 0 \\
(10-t)(5+t) &= 0
\end{aligned}
$$

So $t =$ -5 or 10, so we take $t = 10$ since we want a positive value for time. Then to know how fast the object was traveling when it hits the ground, we substitute $t = 10$ into the derivative. $f'(t) = 5 - 2t$ so $f'(10) = 5 - 2 \cdot 10 = -15$. Hence the object was moving at $\boxed{\text{15m/s}}$ when it hit the ground.

When the object attains its maximum height it has derivative 0, so we find the value of t corresponding to $f'(t) = 0$. $5 - 2t = 0$ implies that $t = \frac{5}{2}$. So the object attains its maximum height at $\boxed{\frac{5}{2}\text{s}}$.

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17. Evaluate the following derivatives:

(a) $\frac{d}{dx}(4e^{x/2})$

(b) $\frac{d}{dx}(\frac{x^2-3}{x})$

(c) $\frac{d}{dx}\left(\frac{2}{\sqrt[5]{x}}\right)$

(d) $\frac{d}{dx}(\frac{x^2-4}{x+2})$

Solution:

(a) $\frac{d}{dx}(4e^{x/2}) = \frac{1}{2} \cdot 4e^{x/2} = \boxed{2e^{x/2}}$

(b) $\dfrac{d}{dx}\left(\dfrac{x^2-3}{x}\right) = \dfrac{d}{dx}\left(\dfrac{x^2}{x} - \dfrac{3}{x}\right) = \dfrac{d}{dx}\left(x - \dfrac{3}{x}\right) = \dfrac{d}{dx}\left(x - 3x^{-1}\right) = 1 + 3x^{-2} = \boxed{1 + \dfrac{3}{x^2}}$

(c) $\dfrac{d}{dx}\left(\dfrac{2}{\sqrt[5]{x}}\right) = \dfrac{d}{dx}\left(2x^{-1/5}\right) = -\dfrac{1}{5} \cdot 2x^{-6/5} = \boxed{-\dfrac{2}{5}x^{-6/5}}$

(d) $\dfrac{d}{dx}\left(\frac{x^2-4}{x+2}\right) = \dfrac{d}{dx}\left(\dfrac{(x+2)(x-2)}{x+2}\right) = \dfrac{d}{dx}(x-2) = \boxed{1}$

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18. Let $f(x) = \frac{1}{x}$. Find an equation for the tangent line to $f(x)$ at $x = 2$. Then use this tangent line to approximate $\frac{1}{3}$. What is the resulting percent error?

Solution:

The slope of this tangent line will be $f'(2)$. Since $f'(x) = -\frac{1}{x^2}$, we get that $f'(2) = -\frac{1}{4}$. The tangent line will also intersect the graph of $f(x)$ at $x = 2$, so it must pass through the point $(2, f(2)) = (2, \frac{1}{2})$. Using point-slope form, we then get the equation:

$$y - \frac{1}{2} = -\frac{1}{4}(x-2)$$
$$\Rightarrow y = -\frac{1}{4}x + \frac{1}{2} + \frac{1}{2}$$
$$\Rightarrow y = -\frac{1}{4}x + 1$$

So the tangent line to $f(x) = \frac{1}{x}$ at $x = 2$ is $\boxed{y = -\frac{1}{4}x + 1}$.

To approximate $\frac{1}{3}$ with the tangent line that is approximating $f(x) = \frac{1}{x}$, we set $x = 3$ to get $y = -\frac{1}{4} \cdot (3) + 1 = -\frac{3}{4} + 1 = \boxed{\frac{1}{4}}$.

The resulting percent error is then:

$\dfrac{\left|\frac{1}{3} - \frac{1}{4}\right|}{\frac{1}{3}} \times 100\% = \dfrac{\frac{1}{12}}{\frac{1}{3}} \times 100\% = \dfrac{1}{12} \cdot \dfrac{3}{1} \times 100\% = \dfrac{1}{4} \times 100\% = 25\%.$

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19. Let $f(x) = -x^3 + 6x^2 - 12x - 5$. For what values of x is $f(x)$ decreasing? For what values of x is $f(x)$ concave down?

Solution:

To determine where $f(x)$ is decreasing, we need to first find its derivative:
$f'(x) = -3x^2 + 12x - 12$
We then set $f'(x)$ to zero to find critical points.

$$\begin{aligned} -3x^2 - 12x + 12 &= 0 \\ \Rightarrow x^2 - 4x + 4 &= 0 \\ \Rightarrow (x-2)(x-2) &= 0 \end{aligned}$$

So there is a critical point at $x = 2$ and therefore we just need to test the sign of $f'(x)$ in the regions $x < 2$ and $x > 2$. $f'(0) = -12 < 0$ and $f'(3) = -3 < 0$ so $f'(x)$ is negative on both intervals. Therefore $f(x)$ is decreasing on both $\boxed{x < 2 \text{ and } x > 2}$.
To determine where $f(x)$ is concave down, we need to find its second derivative:
$f''(x) = -6x + 12$
Setting $f''(x) = 0$ then gives us $-6x + 12 = 0$ so $x = 2$ is our inflection point. So now we need to test the sign of $f''(x)$ in the regions $x < 2$ and $x > 2$. $f''(0) = 12 > 0$ and $f''(3) = -6 < 0$. So $f''(x)$ is negative only on the region $x > 2$. So $f(x)$ is concave down on $\boxed{x > 2}$.

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20. Suppose $T(p)$ represents the time it takes in minutes to get a sandwich at the Arbor Subway when there are p people in line. What does $T(3) = 5$ mean? What does $T'(3) = 1$ mean? What does $T^{-1}(8) = 6$ mean?

Solution:

$T(3) = 5$ means when there are 3 people in line, it takes 5 minutes to get a sandwich.
$T'(3) = 1$ means when there are 3 people in line, the time it takes to get a sandwich is increasing at a rate of 1 minute per person in line.
$T^{-1}(8) = 6$ means it takes 8 minutes to get a sandwich when there are 6 people in line.

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21. Suppose $f(t)$ represents the amount of liters in a water tank after t days. What does $f'(4) = -2$ mean? What does $f''(4) = .5$ mean? If $f''(t)$ is always equal to .5, at what time will the tank no longer be losing water?

Solution:

$f'(4) = -2$ means that at 4 days, the amount of water in the tank is decreasing at a rate of 2 liters/day.
$f''(4) = .5$ means that at 4 days, the rate of water entering/leaving the tank is increasing at a rate of 0.5 liters/day each day (alternatively, it is increasing at a rate of 0.5 liters/day^2).
If $f''(t) = 0.5$ for all t then after each day $f'(t)$ will increase by 0.5. Thus since $f'(4) = -2$, $f'(5)$ will be -1.5, $f'(6)$ will be -1, and so on and then clearly $f'(8) = 0$ and the tank will no longer be losing water. So the tank will no longer be losing water starting on $\boxed{\text{day 8}}$.

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22. Suppose some kids decide to open a lemonade stand on a street corner. It costs them 20 cents to make a single cup of lemonade. After weeks of selling, they find that they can sell 50 cups when they charge \$1 per cup and that their sales fall by 5 for each 10 cents extra that they charge. What price should the kids sell their lemonade to make the most profit? What is the maximum profit they can make?

Solution:

Let n be the number of cups of lemonade the kids sell and p the price for each cup. Then the Profit (P) function for the kids will be $P = np - 0.2n = n(p - 0.2)$ since np represents the amount of income they make and the $-0.2n$ represents the 20 cents cost for making each cup of lemonade. The question asks for the ideal *price* to sell the lemonade, so we want a function for profit in terms of just p.
The fact that the kids sell 50 cups when they charge \$1 per cup gives us a point (1,50) if we graphed p versus n. The fact that sales fall by 5 for each 10 cents extra they charge gives us a slope of $m = -\frac{5}{.1} = -50$.
Point-slope form then gives us:

$$\begin{aligned} n - 50 &= -50(p - 1) \\ \Rightarrow n - 50 &= -50p + 50 \\ \Rightarrow n &= -50p + 50 + 50 \\ \Rightarrow n &= -50p + 100 \end{aligned}$$

Substituting this formula for n into our equation for profit gives us:

$$\begin{aligned} P(p) &= (-50p + 100)(p - 0.2) \\ &= -50p^2 + 100p + 10p - 20 \\ &= -50p^2 + 110p - 20 \end{aligned}$$

So $P'(p) = -100p + 110$ and solving for critical points yields $-100p + 110 = 0 \Rightarrow p = 1.1$. Thus the price that yields the maximum profit is $\boxed{\$1.10}$.
The maximum profit the kids can make then requires substituting this ideal price back into the function for profit. So $P(1.10) = (-50 \cdot 1.10 + 100)(1.10 - 0.2) = (-55 + 100)(0.9) = (45)(0.9) = 40.50$. Thus the kids' maximum profit is $\boxed{\$40.50}$.

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23. Suppose a farmer wants to make a rectangular field that is bordered by a brick wall on one side and wooden fence on the other three sides. The cost of the brick wall is \$5 per meter and the wooden fence costs \$3 per meter. If the farmer only has \$120 to spend, what is the maximum area this field can be?

Solution:

Let L be the length of the brick side of the rectangular perimeter and W denote the other dimension. Then the cost of the entire perimeter would be L meters of brick wall at \$5 per meter and $(2W + L)$ meters of wooden fence at \$3 per meter, yielding Cost $= 5L + 3(2W + L) = 8L + 6W$.
Then since the farmer only has \$120 to spend, we set $8L + 6W = 120$ as we assume the farmer will need to spend all of his money to maximize the area of his field.
Now we want to maximize the area (A) of the field, where $A = WL$. We would like to get a formula for the area in terms of one variable. We can use the above cost formula to do this. $8L + 6W = 120$ implies $W = \dfrac{120 - 8L}{6}$. Substituting this into the area equation gives us:

$A(L) = \left(\dfrac{120 - 8L}{6}\right)L = \dfrac{120L - 8L^2}{6} = 20L - \dfrac{4}{3}L^2$

$\Rightarrow A'(L) = 20 - \dfrac{8}{3}L$

Setting to 0 and solving then yields $20 - \dfrac{8}{3}L = 0 \Rightarrow L = \dfrac{20 \cdot 3}{8} = \dfrac{15}{2}$. Then the maximum area is then:

$$\begin{aligned} A\left(\frac{15}{2}\right) &= 20 \cdot \frac{15}{2} - \frac{4}{3}\left(\frac{15}{2}\right)^2 \\ &= 150 - \frac{4}{3}\left(\frac{225}{4}\right) \\ &= 150 - \frac{225}{3} \\ &= 150 - 75 \\ &= \boxed{75 \text{ m}^2} \end{aligned}$$