High Distance Heegaard Splittings via Dehn Twists

Joint Mathematics Meetings 2013

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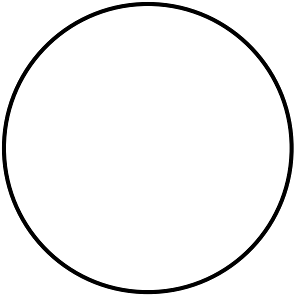
Define terms:

- Heegaard splittings
- Curve complex
- Disk complex
- Hempel distance
- Dehn twists

Heegaard Splittings

Attaching g **handles** to a 3-ball B^3 produces a genus g **handlebody**.



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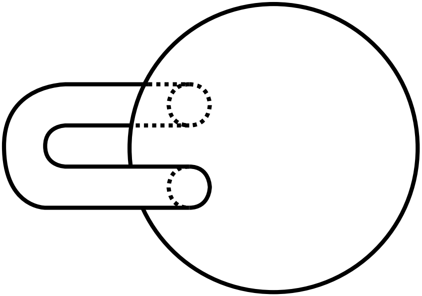

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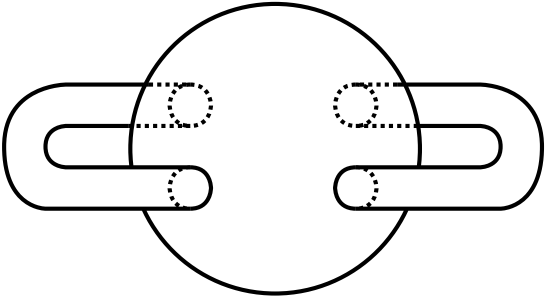

Heegaard Splittings

Let H_1 and H_2 be two (orientable) genus g handlebodies.

- ∂H_1 and ∂H_2 are both closed orientable genus g surfaces and therefore homeomorphic.
- A 3-manifold can be created by attaching H_1 to H_2 by a homeomorphism of their boundaries.

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Definition

The resulting 3-manifold M can be written as $M = H_1 \cup_\Sigma H_2$, $\Sigma = \partial H_1 = \partial H_2$.

*This decomposition of M into two handlebodies of equal genus is called a **Heegaard splitting** of M and Σ is the **splitting surface**.*

Curve Complex

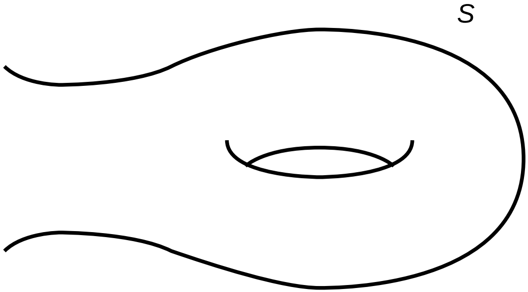
Let S be a closed orientable genus $g \geq 2$ surface.

Definition

*The **curve complex** of S, denoted $\mathcal{C}(S)$, is the following complex:*

- *vertices are the isotopy classes of essential simple closed curves in S*
- *distinct vertices $x_0, x_1, ..., x_k$ determine a k-simplex of $\mathcal{C}(S)$ if they are represented by pairwise disjoint simple closed curves*

Curve Complex



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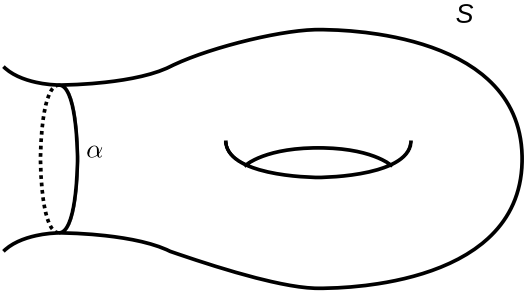

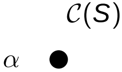

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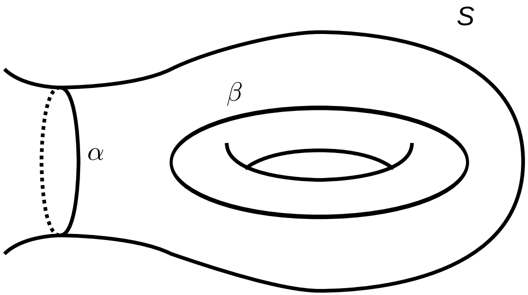

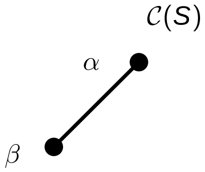

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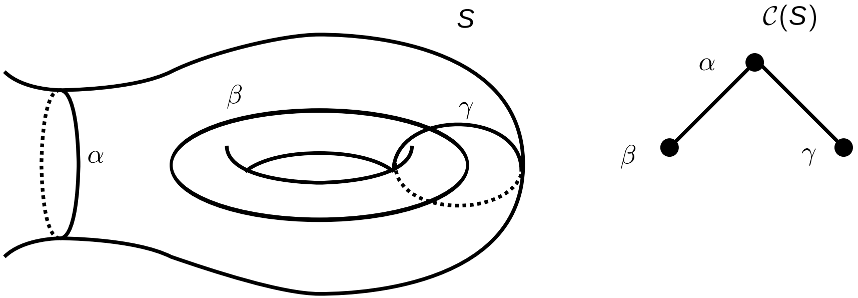


Disk Complex

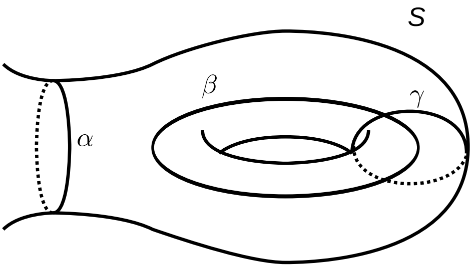
Suppose S is the splitting surface for a Heegaard splitting $M = H_1 \cup_S H_2$.

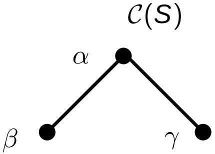
Definition

*The **disk complex** of H_1, denoted $\mathcal{D}(H_1)$ is the subcomplex of $\mathcal{C}(S)$ that bound disks in H_1. Similarly define $\mathcal{D}(H_2)$.*

Disk Complex

Assume embedded in S^3.





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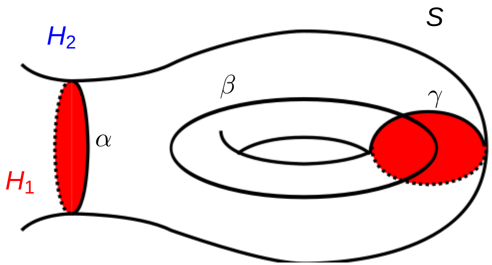

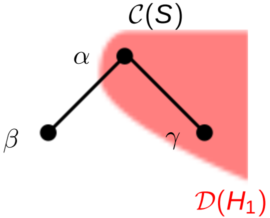

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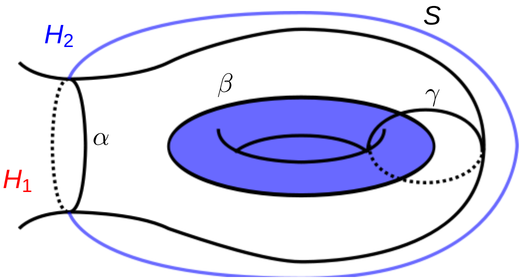

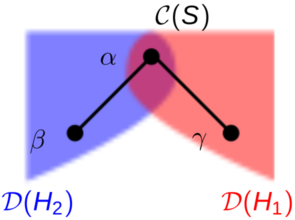

Distance

Definition

*(Hempel, 2001) The **distance** of a splitting $M = H_1 \cup_S H_2$, denoted $d(\mathcal{D}(H_1), \mathcal{D}(H_2))$, is the length of the shortest path in $\mathcal{C}(S)$ connecting $\mathcal{D}(H_1)$ to $\mathcal{D}(H_2)$.*

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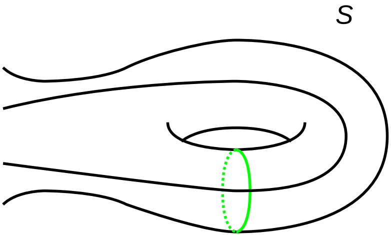
The distance of a splitting can provide information about the original manifold.

- If a manifold admits a distance d splitting, then the minimum genus of an orientable incompressible surface is $d/2$.
- If a manifold admits a distance ≥ 3 splitting, then the manifold has hyperbolic structure.

Dehn twists

A **Dehn twist** is a surface automorphism that can be visualized as a "twist" about a curve on the surface.



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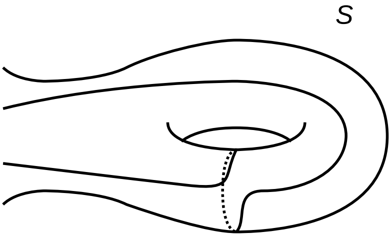

Theorem 1

- H is a genus $g \geq 2$ handlebody,
- γ is a simple closed curve that is distance $d \geq 2$ from $\mathcal{D}(H)$,
- M^k is the 3-manifold created when H is glued to a copy of itself via k Dehn twists about γ.

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(Casson-Gordon, 1987). *For $k \geq 2$, M^k admits a Heegaard splitting of distance ≥ 2.*

Theorem

(Y.,2012). *For $k \geq 2d - 2$, M^k admits a Heegaard splitting of distance exactly $2d - 2$.*

Theorem 2

- H_1 and H_2 are genus g handlebodies with $\partial H_1 = \partial H_2$
- $d(\mathcal{D}(H_1), \mathcal{D}(H_2)) = d_0$
- γ is a simple closed curve that is distance d_1 from $\mathcal{D}(H_1)$ and distance d_2 from $\mathcal{D}(H_1)$
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Theorem

(Casson-Gordon, 1987). *Suppose $d_0 \leq 1$ and $d_1, d_2 \geq 2$. Then for $k \geq 6$, M^k admits a Heegaard splitting of distance ≥ 2.*

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Theorem

(Y.,2012). *Let* $n = \max\{1, d_0\}$. *Suppose* $d_1, d_2 \geq 2$ *and* $d_1 + d_2 - 2 > n$. *Then for* $k \geq n + d_1 + d_2$, M^k *admits a Heegaard splitting of distance at least* $d_1 + d_2 - 2$ *and at most* $d_1 + d_2$.

Thank you!