1. Limits Superior and Inferior

So: most of the definitions and theorems we’ve developed so far for sequences are centered around the concept of convergence – we have lots of ways of talking about when things converge, where they converge to, and under what conditions they will be forced to converge.

However, when we’re confronted with a divergent sequence, it is sometimes useful to be able to say more about it than just “it doesn’t converge!” For example, the sequences

\[
\begin{align*}
0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots \\
1 & 2 1 3 1 4 1 5 1 6 \\
3 & 3 & 4 & 4 & 5 & 5 & 6 & 7 & 7 & 7 & \ldots
\end{align*}
\]

both diverge, and yet both exhibit very clear behaviors at infinity – specifically, both sequences seem to “tend” to both 0 and 1 at infinity. The following definition helps us offer a canonical way of talking about such limiting behaviors at infinity, even when looking at such divergent sequences:

**Definition 1.1.** For a sequence \( \{a_n\} \), set \( x_n = \sup \{a_m : m \geq n\} \). We then define the **limit superior** of \( a_n \) as

\[
\limsup_{n \to \infty} a_n = \lim_{n \to \infty} x_n.
\]

Similarly, if we set \( y_n = \inf \{a_m : m \geq n\} \), we can then define the **limit inferior** of \( a_n \) as

\[
\liminf_{n \to \infty} a_n = \lim_{n \to \infty} y_n.
\]

**Example 1.2.** If \( \{a_n\} \) is the sequence given by

\[
0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots
\]

then for any \( n \) we have that

\[
\begin{align*}
\sup\{a_m : m \geq n\} &= 1, \\
\inf\{a_m : m \geq n\} &= 0,
\end{align*}
\]
and thus that
\[ \limsup_{n \to \infty} a_n = 1, \quad \liminf_{n \to \infty} a_n = 0. \]

**Example 1.3.** If \( \{a_n\} \) is the sequence given by
\[
\begin{align*}
1 & \quad 2 & \quad 1 & \quad 3 & \quad 1 & \quad 4 & \quad 1 & \quad 5 & \quad 1 & \quad 6 \\
3 & \quad 3 & \quad 4 & \quad 4 & \quad 5 & \quad 5 & \quad 6 & \quad 6 & \quad 7 & \quad 7 & \quad \cdots
\end{align*}
\]
then for any \( n \) we have that
\[ 1 \geq \sup \{a_m : m \geq n\} \geq \frac{k-1}{k}, \]
for any sufficiently large \( k \), because every other entry in the sequence \( \{a_n\} \) is of the form \( \frac{k-1}{k} \), where \( k \) increases with \( n \). Consequently, we have that \( \limsup_{n \to \infty} a_n = 1 \). Similarly, for any \( n \) we have that
\[ 0 \leq \inf \{a_m : m \geq n\} < \frac{1}{k}, \]
for any sufficiently large \( k \), and thus that \( \liminf_{n \to \infty} a_n = 0 \).

So: much as with normal limits, we have a number of useful conditions as to when the limits superior and inferior exist. We state and prove these all below:

**Theorem 1.4.** If \( \{a_n\} \) is a bounded sequence, then \( \liminf_{n \to \infty} a_n \) and \( \limsup_{n \to \infty} a_n \) both exist.

**Proof.** Note that for every \( n \), we have that
\[ x_n := \sup \{a_m : m \geq n\} \geq x_{n+1} := \sup \{a_m : m \geq n + 1\}, \]
and thus that the sequence of supremums \( \{x_n\} \) is nonincreasing. As well, because \( \{a_n\} \) is a bounded sequence, the \( x_n \)'s are also bounded; combining these two results then tells us that \( \{x_n\} \) is a convergent sequence – i.e. that \( \limsup_{n \to \infty} a_n \) exists!

Similarly, by noting that
\[ y_n := \inf \{a_m : m \geq n\} \leq y_{n+1} := \inf \{a_m : m \geq n + 1\}, \]
we have that the \( y_n \)'s form a nondecreasing bounded sequence, and thus also converge - i.e. that \( \liminf_{n \to \infty} a_n \) exists as well. \( \square \)

**Theorem 1.5.** \( \lim_{n \to \infty} a_n \) exists if and only if \( \liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n \).

**Proof.** Suppose first that \( \lim_{n \to \infty} a_n \) exists, and furthermore is equal to some \( l \). Then, by definition, we have that for any \( \epsilon > 0 \), there is some \( N \) such that for every \( n > N \), \( a_n \) is within \( \epsilon \) of \( l \).

But this means that (in particular) the biggest of all of the \( a_n \)'s with \( n > N \) has to be within \( \epsilon \) of \( l \) – i.e. that \( \sup \{a_m : m \geq n\} \) must be within \( \epsilon \) of \( l \).

But this is literally the definition of the statement
\[ \lim_{n \to \infty} \sup \{a_m : m \geq n\} = l; \]
which, in other words, is just the statement
\[ \limsup_{n \to \infty} a_n = l. \]
So the limit superior exists. A completely identical argument (just replace the inf’s with sup’s above) shows that the limit inferior exists as well; so we have shown one direction of our “if and only if” statement.

To prove the other direction, assume now that \( \liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n \).

What happens then?

Well – we have, for every \( a_n \), the relation

\[
\inf \{ a_m : m \geq n \} \leq a_n \leq \sup \{ a_m : m \geq n \}.
\]

Thus, by the squeeze theorem for sequences, we have that \( \lim_{n \to \infty} \) exists and is equal to both the \( \limsup \) and \( \liminf \).

\[ \square \]

2. Cauchy Sequences

**Definition 2.1.** We say that a sequence is **Cauchy** if and only if for every \( \epsilon > 0 \) there is a natural number \( N \) such that for every \( m, n \geq N \)

\[ |a_m - a_n| < \epsilon. \]

You can think of this condition as saying that Cauchy sequences “settle down” in the limit – i.e. that if you look at points far along enough on a Cauchy sequence, they all get fairly close to each other.

So: a priori, this definition looks to be somewhat different than our earlier definition of convergence – in particular, the definition of Cauchy makes no reference to any sort of “limit point,” whereas the notion of a sequence having a limit point is key to the idea of convergence. But it seems plausible that these definitions might be equivalent – after all, both conditions are basically ways of saying that a sequence “stops moving at infinity.” This actually turns out to be the case, as the next theorem states:

**Theorem 2.2.** A sequence is Cauchy if and only if it converges.

**Proof.** Suppose that \( \{a_n\} \) is a sequence that converges to some value \( l \); then, by definition, we have that

\[ \forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t } \forall n > N, |a_n - l| < \epsilon. \]

So, in particular, if we pick any \( m, n \) both larger than \( N \), we have that both \( |a_n - l| < \epsilon \) and \( |a_m - l| < \epsilon \). Adding these two results together tells us that

\[
|a_m - l| + |a_n - l| < 2\epsilon \\
\Rightarrow |a_m - l| + |l - a_n| < 2\epsilon \\
\Rightarrow |a_m - l + l - a_n| < 2\epsilon \\
\Rightarrow |a_m - a_n| < 2\epsilon,
\]

(where the above implications follow from the triangle inequality and simple algebraic manipulations.)

But this is precisely the condition that \( \{a_n\} \) is Cauchy! So we have that all convergent sequences are Cauchy.

To see the other direction – suppose that \( \{b_n\} \) is a Cauchy sequence. We seek now to show that this sequence converges.
To do this: notice first that by definition, we have that for every \( \epsilon > 0 \) there is a natural number \( N \) such that for every \( m, n \geq N \)

\[
|b_m - b_n| < \epsilon.
\]

In particular, if we let \( \epsilon = 1 \) and \( n = N \), we have that

\[
\forall m \geq N, |b_m - b_N| < 1 \Rightarrow \forall n \geq N, |b_n| < 1 + |b_N|;
\]

in other words, that the values of \( b_n \) are bounded, for all \( n \geq N \). However, because the set \( \{b_1, b_2, \ldots, b_N\} \) is a finite set, we know that it too is bounded, say by some constant \( C \). By taking the larger of these two bounds, we have that in fact the entire sequence \( \{b_n\} \) is bounded – thus, by our earlier theorems, we know that it has a convergent subsequence! Denote this convergent subsequence by \( \{b_{n_k}\} \), and call the number that it converges to \( l \).

We claim that (in fact) the entire sequence \( \{b_n\} \) converges to \( l \). To see this, we just have to note the following two things:

1. because \( \lim_{k \to \infty} b_{n_k} = l \), we know that
   \[
   \forall \epsilon > 0, \exists N_1 \in \mathbb{N} \text{ s.t } \forall n_k > N_1, |b_{n_k} - l| < \epsilon.
   \]
2. because the sequence \( \{b_n\} \) is Cauchy, we know that
   \[
   \forall \epsilon > 0, \exists N_2 \in \mathbb{N} \text{ s.t } \forall n, m > N_2, |b_m - b_n| < \epsilon.
   \]

So, in particular, if \( N = \max\{N_1, N_2\} \), for every \( n_k, m > N \) we have that \( |b_{n_k} - l| < \epsilon \) and \( |b_m - b_{n_k}| < \epsilon \). Adding these two equations together gives us that

\[
|b_{n_k} - l| + |b_m - b_{n_k}| < 2\epsilon \Rightarrow |b_m - l + b_m - b_{n_k}| < 2\epsilon \Rightarrow |b_m - l| < 2\epsilon,
\]

(where the above implications follow from the triangle inequality and simple algebraic manipulations.)

But this is the definition of convergence! So we have that the sequence \( \{b_n\} \) converges as claimed. \( \square \)

The following example illustrates why the Cauchy criterion is sometimes useful, as it can be much easier to show that certain things are Cauchy than to show that they converge – in particular, if we don’t know where a sequence converges to, it can be much easier to show that it’s Cauchy than to show it converges.

**Example 2.3.** Let

\[
a_n := \sum_{k=1}^{n} \frac{1}{k^2}.
\]

Does \( a_n \) converge?
Proof. So: we claim that \( \{a_n\} \) is in fact Cauchy. To see this: simply notice that for any \( m, n > N \) where \( m > n \)

\[
0 \leq a_m - a_n = \sum_{k=1}^{m} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} = \sum_{k=n+1}^{m} \frac{1}{k^2} < \sum_{k=n+1}^{m} \frac{1}{k(k-1)} = \sum_{k=n+1}^{m} \frac{1}{k-1} - \frac{1}{k} = \sum_{k=n+1}^{m} \frac{1}{k-1} - \sum_{k=n+1}^{m} \frac{1}{k} = \sum_{k=n}^{m-1} \frac{1}{k} - \sum_{k=n+1}^{m} \frac{1}{k} = \frac{1}{n} - \frac{1}{m} < \frac{1}{n} + \frac{1}{m} < \frac{2}{N}.
\]

Thus, if \( N > \frac{2}{\varepsilon} \), we have that for any \( m, n > N \),

\[
|a_m - a_n| < \frac{2}{2/\varepsilon} = \varepsilon.
\]

But this just means that our sequence is Cauchy! So, by the above theorem, it converges. \( \square \)

3. Series and Sums

So: with the above example in mind, we now turn from general sequences to the study of series and infinite sums. We define the concept of an “infinite sum” below:

**Definition 3.1.** A sequence is called **summable** if the sequence \( \{s_n\}_{n=1}^{\infty} \) of partial sums

\[
s_n := a_1 + \ldots + a_n
\]

converges. If it does, we then call the limit of this sequence the **sum** of the \( a_n \), and denote this quantity by writing

\[
\sum_{n=1}^{\infty} a_n.
\]

We first illustrate this definition by a pair of examples:
Example 3.2. For any \( r \in (-1,1) \), does the sum
\[
\sum_{n=0}^{\infty} r^n
\]
converge? If so, what does it converge to?

Proof. So: this can be done by simply performing the following algebraic trick: if we denote the partial sums \( r^0 + r^1 + r^2 + \ldots + r^n \) by the symbols \( s_n \), we have that
\[
\begin{align*}
s_n &= 1 + r^1 + r^2 + \ldots + r^n \\
r \cdot s_n &= r^1 + r^2 + \ldots + r^{n+1} \\
\Rightarrow s_n - r s_n &= 1 - r^{n+1} \\
\Rightarrow s_n(1-r) &= 1 - r^{n+1} \\
\Rightarrow s_n &= \frac{1 - r^{n+1}}{1-r}
\end{align*}
\]

\[\Rightarrow \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{1 - r^{n+1}}{1-r} = \frac{1}{1-r}.\]

But the limit of these partial sums is, by definition, the infinite sum of the \( r^n \)'s! So this sum converges – specifically, it converges to \( \frac{1}{1-r} \). \( \square \)

Example 3.3. Does the sum
\[
\sum_{n=1}^{\infty} \frac{1}{n}
\]
converge?

Proof. This sum, in fact, does not converge! To see this, simply note that if \( s_n \) denotes the partial sums \( 1 + \frac{1}{2} + \ldots + \frac{1}{n} \), we have that
\[
s_n = \sum_{k=1}^{n} \frac{1}{k} \geq \int_{1}^{n} \frac{1}{x} dx = \log(n),
\]
and thus that
\[
\lim_{n \to \infty} s_n \geq \lim_{n \to \infty} \log(n) = \infty,
\]
and so (in particular) does not converge. \( \square \)

So: because series are a special type of sequences, it stands to reason that all of our original theorems on sequences should have a nice interpretation in the language of infinite sums. We describe two such quick results here:

Lemma 3.4. (Cauchy criterion) A sequence \( \{a_n\} \) is summable if and only if
\[
\lim_{m,n \to \infty} |a_n + a_{n+1} + \ldots + a_m| = 0
\]

Proof. Just place the sequence \( \{\sum_{k=1}^{n} a_k\}_{n=1}^{\infty} \) into the definition of a Cauchy sequence, and apply our earlier theorem that says that sequences are Cauchy if and only if they converge. \( \square \)

Theorem 3.5. (Vanishing criterion) If \( \{a_n\} \) is summable, then \( \lim_{n \to \infty} a_n = 0. \)
Proof. Let \( m = n + 1 \) in the limit \( \lim_{m,n \to \infty} |a_n + a_{n+1} + \ldots + a_m| = 0 \), which (by the above) is equivalent to summability. \( \square \)

**Theorem 3.6.** (Boundedness criterion:) Suppose that \( \{a_n\} \) is a sequence of non-negative numbers such that the collection of all partial sums \( \sum_{k=1}^{n} a_n \) is bounded. Then \( \{a_n\} \) is summable.

Proof. So: because the \( a_n \) are all nonnegative, we have that the sequence \( \{\sum_{k=1}^{n} a_n\} \) is nondecreasing. By assumption, it’s also bounded; so it must converge! \( \square \)

These results, admittedly, are pretty trivial, and aren’t really “new” ideas. The following theorem, however, is a tool that’s markedly more powerful than anything we have in the world of just general sequences:

**Theorem 3.7.** (First Comparison Test:) Suppose that \( \{a_n\} \) and \( \{b_n\} \) are sequences such that \( 0 \leq a_n \leq b_n \), for all \( n \). Then \( \{a_n\} \) is summable if \( \{b_n\} \) is summable.

Proof. So: again, because the \( a_n \) are positive, the sequence of partial sums \( \{\sum_{k=1}^{n} a_n\} \) is nondecreasing. As well, because

\[
\sum_{k=1}^{n} a_n \leq \sum_{k=1}^{n} b_n \leq \sum_{k=1}^{\infty} b_n = \text{some fixed constant} \ l < \infty,
\]

the sequence of partial sums is bounded; so it converges! \( \square \)

This test can be thought of as a weirder form of the squeeze theorem – in that to show that a sequence converges, we merely need to show that it is termwise positive and less than another sequence that converges, not that it is pinched by a pair of sequences that converge to the same place! In fact, most of the time when we apply the comparison test, we will be bounding \( \sum_{n=1}^{\infty} a_n \) above by a sum that converges to a value much large than \( \sum_{n=1}^{\infty} a_n \). Some examples follow below:

**Example 3.8.** Let \( a_n = \frac{2}{2^n - \sin(n)} \).

Is \( a_n \) summable?

Proof. Well,

\[
a_n = \frac{2}{2^n - \sin(n)} \leq \frac{2}{2^n - 1} \leq \frac{2}{2^n - 1} = 4 \cdot \frac{1}{2^n},
\]

and \( \sum_{n=1}^{\infty} 4 \cdot \frac{1}{2^n} = 4 \cdot \sum_{n=1}^{\infty} \frac{1}{2^n} = 4 \cdot 1 = 4 \) converges; thus, by the first comparison test, so does \( \sum_{n=1}^{\infty} a_n \). \( \square \)

**Example 3.9.** Let \( a_n = \frac{3n^3}{4n^4 - 1} \).

Is \( a_n \) summable?

Proof. Well,

\[
a_n = \frac{3n^3}{4n^4 - 1} \geq \frac{3n^3}{4n^4} = \frac{3}{4} \cdot \frac{1}{n},
\]

and \( \sum_{n=1}^{\infty} \frac{3}{4} \cdot \frac{1}{n} = \frac{3}{4} \cdot \sum_{n=1}^{\infty} \frac{1}{n} \) diverges; as a result, so does \( \sum_{n=1}^{\infty} a_n \). \( \square \)
As suggested by its name, there is another comparison test beyond the first comparison test, which we describe below:

**Theorem 3.10. (Second Comparison Test:)** Suppose that \( \{a_n\} \) and \( \{b_n\} \) are a pair of positive sequences such that \( \lim_{n \to \infty} \frac{a_n}{b_n} \) exists and is equal to some ratio \( c \neq 0 \). Then \( \{a_n\} \) is summable if and only if \( \{b_n\} \) is summable.

**Proof.** Note first that because \( c \neq 0 \), we have that both the limits \( \lim_{n \to \infty} \frac{a_n}{b_n} = c \) and \( \lim_{n \to \infty} \frac{b_n}{a_n} = \frac{1}{c} \) exist. Thus, without any loss of generality, if we want to show that \( \{a_n\} \) is summable if and only if \( \{b_n\} \) is summable, we can just assume that that \( \{b_n\} \) is summable, and show that \( \{a_n\} \) is consequently summable (as by our argument above, they’re completely interchangeable.)

So: because \( \lim_{n \to \infty} \frac{a_n}{b_n} \) exists, we have (by definition) that there is some \( N \) such that for all \( n > N \),

\[
\frac{a_n}{b_n} \leq 2c
\]

\[
\Rightarrow a_n \leq 2cb_n
\]

So: because the \( a_n \)'s are positive and the sequence \( \sum_{n=1}^{\infty} 2cb_n = 2c \sum_{n=1}^{\infty} b_n \) converges, the first comparison test tells us that the sum \( \sum_{n=1}^{\infty} a_n \) must also converge! This completes our proof. \( \square \)

We conclude with a quick example of the use of this theorem:

**Example 3.11.** Let \( a_n = \frac{(n+1)^3}{4^n} \). Is \( \{a_n\} \) summable?

**Proof.** Well, if \( b_n = \frac{(n)^3}{4^n} \),

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{(n+1)^3}{(n)^3} \frac{4^n}{4^n} = \lim_{n \to \infty} \left( \frac{n+1}{n} \right)^3 = 1;
\]

so, by the second comparison test, \( \{a_n\} \) is summable if and only if \( \{b_n\} \) is. But

\[
\frac{(n)^3}{4^n} \leq \frac{3^n}{4^n} = \left( \frac{3}{4} \right)^n,
\]

which we know to be summable; so by the first comparison test, \( \{b_n\} \) is summable! Consequently, \( \{a_n\} \) is summable as well. \( \square \)