1 Glossary

**Matching** A matching $H$ of a graph $G$ is a 1-regular subgraph of $G$.

**Line Graph** The line graph $L(G)$ of a graph $G$ is the graph with vertex set given by the edges of $G$, and an edge $\{e, f\}$ in $G$ iff these two edges are incident in $G$.

**Edge Coloring** A $n$-edge coloring of a graph $G$ is a mapping from the set $E(G)$ into the set $\{1, 2, \ldots, n\}$ such that no two incident edges receive the same colors.

**Edge Chromatic Number** The edge chromatic number of a graph $G$, $\chi'(G)$, is the smallest value of $n$ such that $G$ admits a $n$-edge coloring.

2 Hall’s Marriage Theorem

**Theorem 1** Take a bipartite graph $G = (A, B)$. Then, the following conditions are equivalent:

- $G$ has a 1-factor.
- (Hall’s condition): For any subset $X \subset A$ or $X \subset B$, if $N(X)$ denotes the neighbors of $X$, then $|X| \leq |N(X)|$.

**Proof.** ($\Rightarrow$): Suppose that $G$ has a 1-factor; because $G$ is bipartite, such a 1-factor is just a pairing-up of vertices in $A$ and in $B$ along edges in $G$. Thus, for any subset $X \subset A$, because $N(X)$ must contain the edges in this 1-factor, we have that $|X| \leq N(X)$ (and similarly for $X \subset B$.)

($\Leftarrow$): Take any matching $M$ in $G$. Consider the following algorithm for creating an alternating path between distinct vertices in $A$ and $B$:

1. Suppose without loss of generality that $M$’s not already a 1-factor, and pick some $a_0 \in A$ that’s nor involved in $M$.

2. Suppose that the sequence $a_0b_1a_1b_2a_2\ldots b_{k-1}a_{k-1}$ has been created. Then, because the set $\{a_0\ldots a_{k-1}\}$ of chosen $A$-vertices is strictly larger than the set $\{b_1\ldots b_{k-1}\}$ of chosen $B$-vertices, there must be some element $b \in B$ that’s connected by some edge $\{b, a\}$ to some previously-chosen $a_i$, by Hall’s condition. Let $b_k$ be equal to $b$, and define $f(k) = i$ (so that $\{b_k, a_{f(k)}\}$ is the edge we used here to pick $b$.)
3. If \( b_k \) is in \( M \), let \( a_k \) be the vertex across from \( b_k \) in \( M \), and return to (2) to continue to grow our sequence. Otherwise, end our sequence! By construction, we know that \( b_k \) is unique amongst the previously chosen \( b_i \)'s; similarly, because we picked the \( a_i \) up to this point by using the matching \( M \), we know that they’re all distinct. So this is still a sequence of distinct vertices!

4. Let the sequence that this algorithm terminates with be denoted as \( a_0b_1\ldots b_{k-1}a_{k-1}b_k \). Notice, now, that this sequence of vertices, by construction, have the following properties:
   - \( a_0 \) and \( b_k \) are both unmatched.
   - \( b_i \) is adjacent to some element in \( \{a_0\ldots a_{i-1}\} \).
   - \( a_i b_i \) is in \( M \), for all \( i \).

5. So: consider the following path, made by alternately following the edges of \( M \) and the edges recorded by the function \( f \):

\[
bb_k, \{b_k, a_f(k)\}, b_f(k), \{a_f(k), b_f(k)\}, b_f(k), \{b_f(k), a_{f^2(k)}\}, a_{f^2(k)}, \ldots, \{b_{f^n(k)}, a_{f^{n+1}(k)}\},
\]

where \( a_{f^{n+1}(k)} = a_0 \).

All of the edges \( \{a_f(k), b_f(k)\} \) lie in \( M \), while none of the edges \( \{b_k, a_f(k)\} \) lie in \( M \), by construction. So, replace the collection of \( \{a_f(k), b_f(k)\} \) in \( M \) with the collection of \( \{b_k, a_f(k)\} \) edges! This collection has precisely one more edge than the old collection, and only deals with the vertices \( a_0, b_k \) (which weren’t in \( M \) anyways) and \( a_1\ldots a_{k-1}, b_1\ldots b_{k-1} \) (which were involved in edges we removed from \( M \) – so it preserves \( M \)'s status as a matching! Thus, repeating this process will allow us to grow \( M \) into a 1-factor.

**Corollary 2** A \( k \)-regular bipartite graph \( G = (A,B) \) can be decomposed into \( k \) disjoint 1-factors.

**Proof.** Pick any subset \( X \subset A \) or \( X \subset B \) of size \( n \). Because \( G \) is \( k \)-regular, there are \( kn \) distinct edges leaving \( X \) and entering \( N(X) \). Consequently, as each vertex has degree \( k \), there must be at least \( n \) vertices in \( N(X) \) to absorb these edges! – so \( |N(X)| \geq |X| \).

Thus, by Hall’s Marriage theorem, there is a 1-factor in \( G \). Deleting it from \( G \) leaves a \( k - 1 \)-regular graph; so repeating this process leaves us with a decomposition of \( G \) into \( k \) distinct 1-factors.

### 3 Edge Colorings

**Proposition 3** A cycle \( C_n \) has edge-chromatic number \( \chi'(G) = \chi(G) \).

**Proof.** Take a cycle \( C_n \), and consider its line graph \( L(C_n) \). This is another cycle! In fact, it’s the same cycle as \( G \), as it has the same number of vertices; thus, its edge chromatic number is the same as \( G \).
**Theorem 4** If $G = (A, B)$ is a bipartite graph, then $\chi'(G) = \Delta(G)$.

**Proof.** As we showed earlier in lecture, a $k$-regular bipartite graph $G$ can be decomposed into $k$ disjoint 1-factors. Simply coloring each of these 1-factors a different color, then, will insure that we have a $k$-edge-coloring of $G$, as no 1-factor contains two incident edges (by definition.)

So, it suffices to show that we can embed any bipartite graph $G$ with maximum degree $\Delta(G)$ as a subgraph of some $\Delta(G)$-regular bipartite graph (as a $k$-edge coloring of a graph gives, by restriction, a $k$-edge coloring of all of its subgraphs.) To do this,

- simply add vertices to either $A$ or $B$ so that both sides have the same number of vertices, and then
- take any vertex $a \in A$ that doesn’t have degree $\Delta(G)$. Then, because the number of edges leaving $A$ is the same as the number of edges entering $B$, and all vertices have degree $\leq \Delta(G)$, there must be some vertex $b$ in $B$ also with degree $< \Delta(G)$. Add an edge between these two vertices! Repeat this process until the graph is $\Delta(G)$-regular.