Perfect Graph Theory	Instructor: Padraic Bartlett

Lecture 3: Chordal Graphs

Week 1 *Mathcamp 2011*

When we defined perfect graphs in our last lecture, the idea was that they would be graphs with "easy-to-determine" chromatic numbers. Has this been true so far?

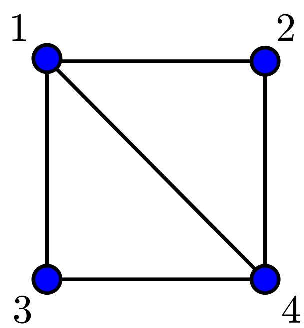
For the three classes of graphs we've shown to be perfect: yes! Complete graphs have trivial-to-find chromatic numbers, bipartite graphs also have easily-found chromatic numbers, and so do their line graphs (the chromatic number of the line graph $L(G)$ of a bipartite graph G is $\Delta(G)$, which we proved in our last lecture.)

Motivated by this success, our lecture today will invert this process: first, we'll try to find a family of graphs that (if they were perfect) would have an easily-calculated chromatic number, and then we'll see if this means they're actually perfect after all!

1 Perfect Elimination Orderings

Definition. In a graph G, a vertex v is called **simplicial** if and only if the subgraph of G induced by the vertex set $\{v\} \cup N(v)$ is a complete graph.

For example, in the graph below, vertex 3 is simplicial, while vertex 4 is not:



A graph G on n vertices is said to have a **perfect elimination ordering** if and only if there is an ordering $\{v_1, \ldots v_n\}$ of G's vertices, such that each v_i is simplicial in the subgraph induced by the vertices $\{v_1, \ldots v_i\}$. As an example, the graph above has a perfect elimination ordering, witnessed by the ordering $(2, 1, 3, 4)$ of its vertices.

Why do we mention this definition? Well: if a graph G admits a perfect elimination ordering, then we have a really fast way to find its clique number: just look at the n different cliques

- $(\{v_n\} \cup N(v_n)) \cap \{v_1, \ldots v_n\}$,
- $(\{v_{n-1}\} \cup N(v_{n-1})) \cap \{v_1, \ldots v_{n-1}\}$,
- ...

- $\{v_1\}$.

If H is an induced subgraph corresponding to a maximum-size clique in our graph, and $v_k \in V(H)$ is the vertex in H with the largest subscript value in our ordering, then by definition $H = (v_k \cup N(v_k)) \cap \{v_1, \ldots v_k\}$; therefore, H comes up in our list! So, to find the largest clique, we just have to check n different graphs. This stands in sharp contrast to the normal situation for graphs, where finding $\omega(G)$ is a NP-complete problem.

In a very well-defined sense, then, we've shown that graphs that have perfect elimination ordering are graphs that would have really easy to find chromatic numbers (if they were perfect!) So: are they?

As it turns out: yes! We prove this here, in two propositions:

Proposition 1 *If G admits a perfect elimination ordering, so do any of its induced subgraphs.*

Proof. Let $\{v_1, \ldots v_n\}$ be G's perfect elimination ordering, $i_1 < i_2 < \ldots i_k$ be any subsequence of the sequence $\{1, 2, \ldots n\}$, and H the corresponding induced subgraph of G on $\{v_{i_1}, \ldots v_{i_k}\}$. By definition, we had that each of the graphs

- $(\{v_{i_k}\} \cup N(v_n)) \cap \{v_1, \ldots v_{i_k}\}$,
- $(\{v_{i_{k-1}}\} \cup N(v_{n-1})) \cap \{v_1, \ldots v_{i_{k-1}}\}$,
- $\ldots$
- $\{v_{i_1}\} \cup N(v_{i_1})) \cap \{v_1, \ldots v_{i_1}\}$

were cliques in G; therefore, by restricting to H, we have that all of the sets

- $(\{v_{i_k}\} \cup N(v_n)) \cap \{v_{i_1}, \ldots v_{i_k}\}$,
- $(\{v_{i_{k-1}}\} \cup N(v_{n-1})) \cap \{v_{i_1}, \ldots v_{i_{k-1}}\}$,
- $\ldots$
- $\{v_{i_1}\}$

are still cliques. Therefore, this induced subgraph H still admits a perfect elimination ordering.

Proposition 2 *If G admits a perfect elimination ordering, G is perfect.*

Proof. By our above proposition, it suffices to just show that $\chi(G) = \omega(G)$ for any graph G with a simplicial elimination ordering. We proceed by induction on the number of vertices in G. If $|V(G)| = 1$, G is trivially perfect, as it's K_1.

Assume now that $V(G) = n > 1$, for some n, and let $\{v_1, \ldots v_n\}$ be the perfect elimination ordering of G's vertices that we're given. Look at the graph $G \setminus \{v_n\}$, formed by deleting v_n from G. By our proposition, $G \setminus \{v_n\}$ still admits a simplicial elimination ordering. Therefore, we can apply our inductive hypothesis to see that $G \setminus \{v_n\}$ is perfect: i.e. that $\chi(G \setminus \{v_n\}) = \omega(G \setminus \{v_n\})$.

For brevity's sake, define $k = \omega(G \setminus \{v_n\})$. In G itself, by definition, we know that the collection of vertices $v_n \cup N(v_n)$ induces a clique as a subgraph: therefore, we know that $N(v_n)$ itself induces a clique, and therefore that $\deg(v_n) = |N(v_n)| \leq \omega(G \setminus \{v_n\}) = k$. So v_n has less than k neighbors.

Suppose that $\deg(v_n) < k$. Then, given any k-coloring of $G \setminus \{v_n\}$, we can extend it to a coloring of G by just letting v_n be whatever color in $\{1, \ldots k\}$ doesn't show up in its neighbors. This means that $\chi(G) = k = \omega(G \setminus \{v_n\}) \leq \omega(G)$, and therefore that G is perfect.

Conversely, assume that $\deg(v_n) = k$. Then $v_n \cup N(v_n)$ forms a clique of size $k + 1$, so $\omega(G) \geq k+1$. Finally, because $\chi(G \setminus \{v_n\}) = k$, we can extend any k-coloring of $G \setminus \{v_n\}$ to a $k+1$-coloring of G by painting v_n the color $k+1$; this shows that $\chi(G) \leq k+1 \leq \omega(G)$, and therefore (again) that G is perfect.

Excellent! The only somewhat unsatisfying part of this new family of graphs is that their property – this perfect elimination ordering – is a kind of ponderous thing, and not quite as obviously easy to check as (say) being bipartite, or being the line graph of a bipartite graph. One of the other motivations we had for defining perfect graphs was our hope that it would lead us to a "nice" characterizing property, similar to the one we had for bipartite graphs; does one exist for these "perfect elimination ordering" graphs?

As it turns out, yes!

2 Chordal Graphs

Definition. A graph G is said to contain a **chordless cycle** if and only if it has some induced subgraph isomorphic to a cycle C_t, for $t \geq 4$. If a graph does not contain any chordless cycles, it is called **chordal**.

Definition. For any two vertices $x, y \in G$ such that $\{x, y\} \notin E(G)$, a $x - y$ **separator** is a set $S \subset V(G)$ such that the graph $G \setminus S$ has at least two disjoint connected components, one of which contains x and another of which contains y.

Theorem 3 *For a graph G on n vertices, the following conditions are equivalent:*

1. *G has a perfect elimination ordering.*
2. *G is chordal.*
3. *If H is any induced subgraph of G and S is a vertex separator of H of minimal size, S's vertices induce a clique.*

Proof. ($1 \Rightarrow 2$:) Let C be any cycle in G of length at least 4. Take our perfect elimination ordering of G, and start deleting vertices according to this ordering until you get to an element c in C. When you delete this element in C, we know that its neighbors in C have to induce a clique: therefore, there is a "chord" (i.e. edge) between two elements in C, and therefore the induced subgraph on the vertices in C is not a cycle.

($2 \Rightarrow 3$:) Any induced subgraph of a chordal graph is chordal, because any cycle in G has a chord in it, which will be preserved in any induced subgraphs containing that cycle.

So it suffices to prove that if G is chordal, any minimal $x - y$ separator S will induce a clique.

To do this: let S be a minimal $x-y$ separator in G, and let A_x, A_y be the two connected components of G that contain x and y, respectively. Suppose that u, v are a pair of vertices in S; then, because S is minimal, there are edges from both u and v to the two components A_x, A_y (otherwise, we wouldn't have needed them to separate A_x from A_y. Let P_x be the shortest path from u to v in A_x, and P_y be the shortes path from u to v in A_y; because both of these paths have length ≥ 2, their union is a cycle of length ≥ 4. Because G is assumed to be chordal, there must be a chord in this cycle; because there are no direct edges from A_x to A_y (because they're distinct connected components when we cut along S) nor any other edges from u, v to these components (because we picked shortest-possible paths), the only possible chord can be if $\{u, v\}$ itself is an edge! Because this holds for every pair of vertices $u, v \in S$, we have that there is an edge between every pair of vertices in S: i.e. S induces a clique.

($3 \Rightarrow 1$:) We proceed by induction on n, the number of vertices in G.

In particular, we will prove the following second claim by induction: given any chordal graph G, then either G is a complete graph or G has two vertices x, y that (1) do not share an edge and (2) are both simplicial. If we have this, then note that we immediately have that G has a perfect elimination ordering: simply take a simplicial vertex in G, delete it, and note that induction gives us that this smaller chordal graph has a perfect elimination ordering. Adding our vertex back in does not change the existence of a perfect elimination ordering, so we have proven our claim!

For $n = 1$ this is trivial. Inductively, assume that our claim holds for all graphs on $\leq n-1$ vertices, and seek to prove it for graphs on n vertices. If G is a clique, we are trivially done, as any ordering of G's vertices gives a perfect elimination ordering. Otherwise, there are a pair of vertices x, y such that $\{x, y\}$ is not an edge in $E(G)$. Let S be a minimal $x - y$ separator, and A_x, A_y the components of $G \setminus S$ containing x and y. There are two possibilities: either both of x, y are simplicial (in which case we're done!) or one (or both) are not simplicial.

Assume that x is not simplicial for now. In this case, look at the subgraph induced by the vertices in $S \cup A_x$. This is a proper induced subgraph of our graph, and thus (by induction, as induced subgraphs of chordal graphs are chordal) contains two simplicial vertices with no edge between them. At least one of these simplicial vertices is in A_x, because there is no edge between these two simplicial vertices. Moreover, the only edges this vertex has are to elements of A_x and S, so this vertex is in fact simplicial in our larger overall graph, and not connected to any vertex in A_y!

Applying this logic to y in the event that y is not simplicial gives us a simplicial vertex in A_y as well, and thereby proves our claim.

Perfect elimination graphs, therefore, are chordal – a remarkably elegant classification!