Math 108a Professor: Padraic Bartlett

Homework 1: Fields

Due Thursday, Oct. 3, 3pm, South Hall 6516 *UCSB 2013*

Remember: homework problems need to show work in order to receive full credit. Simply stating an answer is only half of the problem in mathematics; you also need to include an argument that persuades your audience that your answer is correct! As always, if you have any questions, feel free to contact either Shahab or I via email or office hours. Have fun!

1. Consider the following potentially "new" property that we didn't list in class for the real numbers:

   - **New property**?$(+)$: $\forall a, b \in \mathbb{R}$, if both a and $b \neq 0$, then $a + b \neq a, b$.

   Using **only** the field axioms as discussed in class, prove that this property holds for the real numbers. (This is like what we did in class: we are showing that while this property may look like one we have not listed, it is a necessary consequence of the field axioms, and is therefore true in every field.)

2. In lecture, we said that $\mathbb{C}$ satisfied the field axioms, but didn't include proofs that $\mathbb{C}$ satisfies all of these properties.

   Fix this. Specifically, using the definition of $\mathbb{C}$, mimic the proofs that we developed for $\mathbb{Q}$ to show that $\mathbb{C}$ satisfies the **associativity**$(+)$ and **commutativity**$(\cdot)$ properties. (If you want, you can prove all of the other axioms that we didn't prove in class, but we're just looking for these two in your homework.)

3. Consider the complex number $z = \dfrac{1 + i\sqrt{3}}{2}$. Show that this number is a "sixth root" of 1: i.e. that $z^6 = 1$.

4. Show that $\mathbb{Z}/2\mathbb{Z}$ is a field.

5. Show that $\mathbb{Z}/9\mathbb{Z}$ is **not** a field.

6. Find a set S along with operations $+, \cdot$ such that every nonzero element in $\langle S, +, \cdot \rangle$ has a multiplicative inverse, but that is **not** a field.

7. Let $z \in \mathbb{C}$ be a complex number. Show that if $z = \overline{z}$, then z is also a real number: i.e. that its imaginary component is zero.

8. How long did you spend on this set? (This question is just for calibration purposes, and will not change your score or be in any way attached to your name.)