Math 108a	Professor: Padraic Bartlett

Homework 4: Properties of Vector Spaces

Due Thursday, Oct. 24, 3pm, South Hall 6516 — *UCSB 2013*

Remember: homework problems need to show work in order to receive full credit. Simply stating an answer is only half of the problem in mathematics; you also need to include an argument that persuades your audience that your answer is correct! As always, if you have any questions, feel free to contact either Shahab or I via email or office hours. Have fun!

There is also a bonus question! It is completely optional. Completing it will give you ten free points. (It's really hard.)

1. Consider the following map $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_4(\mathbb{R})$:
$$T(a + bx + cx^2) = \int_0^x \left(a + bt + ct^2\right) dt.$$
Show that this a linear transformation. Find the range and null space of this map.

2. Consider the following map $T : \mathbb{R}^3 \to \mathcal{P}_1(\mathbb{R})$:
$$T(a, b, c) = cx - \frac{d}{dx}\left(a + bx^2\right)$$
Show that this a linear transformation. Find the range and null space of this map.

3. Take any one-dimensional vector space[1] V over a field F. Let T be a linear map $V \to V$. Show that there is some constant $\lambda \in F$ such that for any $\vec{v} \in V$, $T(\vec{v}) = \lambda\vec{v}$.

4. Let $T : U \to V$ be a linear map, and let $T(\vec{0}) = \vec{v}$. What are the possible values of $\vec{v}$?

5. Suppose that $T : U \to V$ is a linear map, and that $S : V \to W$ is also a linear map. Consider the map $S \circ T : U \to W$, defined by
$$S \circ T(\vec{u}) = S(T(\vec{u})).$$
Is this a linear map? Why or why not?

6. Can you find a linear map $T : \mathbb{R}^3 \to \mathbb{R}$ such that
$$\text{null}(T) = \{(x, x, x) \mid x \in \mathbb{R}\}?$$
Either construct such a map if it exists, or prove that it cannot exist.

7. How long did this set take you? (As always, asked for calibration purposes.)

Bonus! Find a map $T : \mathbb{R} \to \mathbb{R}$ that is additive, but not homogeneous. Or prove that no such map exists.

[1] Common examples are $\mathbb{R}$ and $\mathbb{C}$, though there are others. Your proof should work for general one-dimensional vector spaces, not just for these two examples.