Math 108a	Professor: Padraic Bartlett

Practice Homework 5: Injection, Surjection, and Linear Maps

Not due: just for practice — *UCSB 2013*

Have fun!

1. Let $T : U \to V$ and $S : V \to W$ be a pair of injective maps. Define the composition of these two maps $S \circ T : U \to W$ as the map $S \circ T(\vec{u}) = S(T(\vec{u}))$.

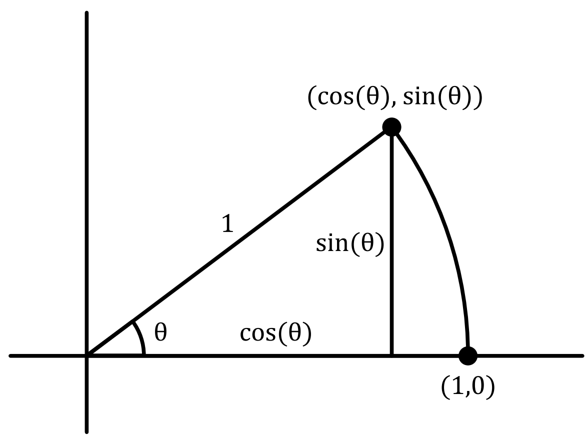
   Is $S \circ T$ injective? Either prove that it is injective, or construct a counterexample.

2. Let $T : U \to V$ and $S : V \to W$ be a pair of surjective maps. Is $S \circ T$ necessarily surjective? Either prove that it must be surjective, or construct a counterexample.

3. Let $S = \{\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}\}$ be some set of vectors, and let $\text{span}(S) = V$ denote the vector space spanned by these vectors. Let $T : V \to V$ be a surjective linear map. Show that $T(S) = \{T(\vec{v_1}), T(\vec{v_2}), \ldots T(\vec{v_n})\}$ also spans V.

4. Let $T : \mathbb{C}^3 \to \mathbb{C}^2$ be a linear map with nullspace $\text{null}(T) = \{(z_1, z_1 + z_2, z_2) \mid z_1, z_2 \in \mathbb{C}\}$. Can T be surjective? Either prove that T cannot be surjective, or find an example of such a map T that is surjective.

5. Consider the map $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$, that takes a vector (x, y) and rotates it by angle θ in a counterclockwise direction around the origin. For example, the vector $(1, 0)$ gets mapped to $(\cos(\theta), \sin(\theta))$, as depicted below:



   (a) Show that the map T_θ is linear.

   (b) Find coefficients $\alpha, \beta, \gamma, \delta$ such that

$$T(x, y) = (\alpha x + \beta y, \gamma x + \delta y).$$