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| Math 108a | Professor: Padraic Bartlett |

# Homework 7: Matrices

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| *Due Friday, 11/15/13, 1:30 pm* | *UCSB 2013* |

This set is shorter, because this week is shorter (no class Monday.) It deals with the concept of matrices. We'll talk about these in class on Wednesday, but if you want a quick preview:

**Definition.** Take a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$. Let the vectors $\vec{e_1}, \ldots \vec{e_n}$ denote the standard basis vectors for $\mathbb{R}^n$: i.e. $\vec{e_1} = (1, 0, \ldots 0), \vec{e_2} = (0, 1, 0 \ldots 0), \ldots \vec{e_n} = (0, 0 \ldots 0, 1)$.

For each of the vectors $T(\vec{e_i})$ in $\mathbb{R}^m$, write

$$T(\vec{e_i}) = (t_{1,i}, t_{2,i}, \ldots, t_{m,i}),$$

where the values $t_{i,j}$ are all real numbers

We can turn $T$ into an $m \times n$ **matrix**, i.e. a $m \times n$ grid of real numbers, as follows:

$$T \longrightarrow T_{\text{matrix}} = \begin{bmatrix} t_{1,1} & t_{1,2} & \ldots & t_{1,n} \\ t_{2,1} & t_{2,2} & \ldots & t_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{m,1} & t_{m,2} & \ldots & t_{m,n} \end{bmatrix}.$$

In other words,

$$T \longrightarrow T_{\text{matrix}} = \begin{bmatrix} \vdots & \vdots & \ldots & \vdots \\ T(\vec{e_1}) & T(\vec{e_2}) & \ldots & T(\vec{e_n}) \\ \vdots & \vdots & \ldots & \vdots \end{bmatrix},$$

Similarly, given some $m \times n$ matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} \end{bmatrix},$$

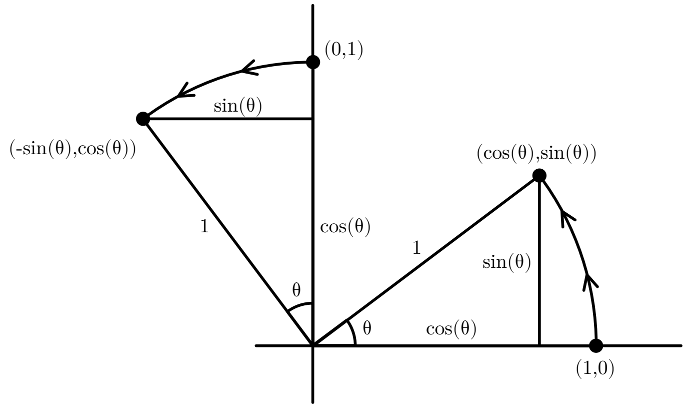
we can interpret $A$ as a **linear map** $A_{\text{map}} : \mathbb{R}^n$ to $\mathbb{R}^m$ as follows:

- For any of the standard basis vectors $\vec{e_i}$, we define $A_{\text{map}}(\vec{e_i})$ to simply be the vector $(a_{1,i}, \ldots a_{m,i})$.
- For any other vector $(x_1, \ldots x_n) \in \mathbb{R}^n$, we define $A_{\text{map}}(x_1, \ldots x_n)$ to simply be the corrresponding linear combination of the $\vec{e_i}$'s: i.e.

$$A_{\text{map}} : (x_1, \ldots x_n) := x_1 \cdot A_{\text{map}}(\vec{e_1}) + \ldots + x_n A_{\text{map}}(\vec{e_n}).$$

In practice, we will usually not bother writing the subscripts "map" and "matrix" on these objects, and think of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $m \times n$ matrices as basically the same things.

For example, consider the map $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$, that we worked with on problem #6 of the homework.




Because this map sends $(1,0)$ to $(\cos(\theta), \sin(\theta))$, and $(0,1)$ to $(-\sin(\theta), \cos(\theta))$, we would express this map as a matrix as follows:

$$T_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Under this interpretation, we would say that $T_\theta(x,y) = (x\cos(\theta) - y\sin(\theta), x\sin(\theta + y\cos(\theta))$, i.e.

$$T_\theta(x,y) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = (x\cos(\theta) - y\sin(\theta), x\sin(\theta + y\cos(\theta))$$

This problem set asks you a few questions about matrices. As always, problems need to show work and proofs in order to receive full credit. Have fun!

1. Write the following linear maps as matrices:

   (a) $T : \mathbb{R}^6 \to \mathbb{R}^6$, defined such that
   $$T(u,v,w,x,y,z) = (z,y,x,w,v,u).$$

   (b) $T : \mathbb{R}^n \to \mathbb{R}^{n-1}$, defined such that
   $$T(x_1, \ldots x_n) = (x_2, x_3, \ldots x_n).$$

   (c) $S^{-1} \circ T \circ S : \mathbb{R}^4 \to \mathbb{R}^4$, where $S : \mathbb{R}^4 \to \mathcal{P}_3(\mathbb{R})$, $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ are the maps
   $$S(a,b,c,d) = (a + bx + cx^2 + dx^3), \text{ and}$$
   $$T\left(a + bx + cx^2 + dx^3\right) = \frac{d}{dx}\left(a + bx + cx^2 + dx^3\right).$$

2. Write the following linear maps as matrices:

   (a) $R \circ T \circ S : \mathbb{R}^4 \to \mathbb{R}^5$, where $S : \mathbb{R}^4 \to \mathcal{P}_3(\mathbb{R})$, $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_4(\mathbb{R})$ and $R : \mathcal{P}_4(\mathbb{R}) \to \mathbb{R}^5$ are the maps

$$\begin{aligned} S(a,b,c,d) &= (a + bx + cx^2 + dx^3), \\ T\left(a + bx + cx^2 + dx^3\right) &= \int_0^x (a + bt + ct^2 + dt^3)dt, \text{ and} \\ R\left(a + bx + cx^2 + dx^3 + ex^4\right) &= (a,b,c,d,e). \end{aligned}$$

   (b) $T : \mathbb{R}^n \to \mathbb{R}^n$, defined such that

$$T(\vec{x}) = \vec{x}.$$

   (c) $T : \mathbb{R}^4 \to \mathbb{R}^4$, defined such that

$$T(w,x,y,z) = (w, w+x, w+x+y, w+x+y+z).$$

3. Take any pair of linear maps $A, B : \mathbb{R}^3 \to \mathbb{R}^3$ with associated matrices

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}, B = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{bmatrix}.$$

   Consider the linear map $B \circ A$ given by composing these two linear maps together.

   (a) Calculate $B \circ A$ when it is applied to the three vectors $\vec{e_1} = (1,0,0), \vec{e_2} = (0,1,0), \vec{e_3} = (0,0,1)$.

   (b) Write out the matrix corresponding to $B \circ A$.