Math 108a	Professor: Padraic Bartlett
Lecture 10: Range and Null Space, part 2	
Week 4	*UCSB 2013*

In our last few classes, we've discussed the two concepts of **range** and **null space**. In particular, the last set of notes that we put online ended with a discussion about **why** we care about these two concepts. We continue that discussion in this class.

1 Range and Null Space: Motivational Example

In our last class, we said that the motivation for studying the **range** of a linear map was not too difficult to understand: given a linear map T, we often would want to understand what sorts of objects can be created as outputs of that map.

The null space was odder. On one hand, understanding the collection of all things that goes to 0 seems somewhat silly; why do we care so much about 0? Why not any other value?

To understand this, we considered the map

$$T : \mathbb{R}^3 \to \mathbb{R},$$

defined by

$$T(x, y, z) = x + y.$$

The null space of this map is just the collection of all triples (x, y, z) such that

$$T(x, y, z) = 0;$$

i.e. it's the set

$$\text{null}(T) = \{(x, -x, z) : x, z \in \mathbb{R}\}.$$

So, here's a related question. What does the set of all vectors that map to 1 look like? Well, if we directly solve, we're looking for all triples (x, y, z) such that

$$T(x, y, z) = 1;$$

i.e. it's the set

$$\{(1 + x, -x, z) : x, z \in \mathbb{R}\}.$$

In other words, it's basically what happens if we take null(T) and scale every element in it by $(1, 0, 0)$!

Furthermore, if we take **any** real number $a \in \mathbb{R}$. we can see that

$$T(x, y, z) = a$$

if and only if our triple has the form

$$(a+x,-x,z),$$

for some $x, z \in \mathbb{R}$.

So, in a sense, when we understood the null space of the linear map T above, for any a we understood the collection of **all** elements that map to that a! So there's nothing special about 0, in a sense — rather, the null space appears to be capturing the total "redundancy" of our map, i.e. the number of elements that our maps sends to **any** element!

We make this rigorous with the following theorem:

Theorem 1. *Let $T: U \to V$ be a linear map. Let $N(T)$ denote the null space of T, and $\vec{u}, \vec{w}$ be any pair of vectors from U, V respectively such that $T(\vec{u}) = \vec{v}$.*

Let $T^{-1}(\vec{v})$ denote the set of all vectors in U that get mapped to $\vec{v}$ by T: i.e.

$$A_{\vec{v}} = \{\vec{w} \in U \mid T(\vec{w}) = \vec{v}\}.$$

Then $T^{-1}(\vec{v})$ is just $N(T)$ translated by $\vec{u}$! In other words,

$$T^{-1}(\vec{v}) = \{\vec{w} \in U \mid \textit{there is some } \vec{x} \in N(T) \textit{ such that } \vec{w} = \vec{x} + \vec{u}\}$$

In other words, understanding the collection of elements that all get mapped to $\vec{0}$ basically lets us understand the collection of elements that get mapped to any fixed vector $\vec{v}$.

Proof. Let $\vec{u}, \vec{w}$ be any pair of vectors from U, V respectively such that $T(\vec{u}) = \vec{v}$.

Take any vector $\vec{w} \in T^{-1}(\vec{v})$. By definition, we know that $T(\vec{w}) = \vec{v}$.

Look at the vector $\vec{w} - \vec{u}$. If we use the fact that T is linear, we can see that

$$T(\vec{w} - \vec{u}) = T(\vec{w}) - T(\vec{u}) = \vec{v} - \vec{v} = \vec{0};$$

therefore, $\vec{w} - \vec{u}$ is in the null space $N(T)$ of T. Therefore, we can write

$$\vec{w} = (\vec{w} - \vec{u}) + \vec{u};$$

i.e. we can write $\vec{w}$ as the sum of an element from $N(T)$ and the vector $\vec{u}$.

Now, take any vector $\vec{x} \in N(T)$. Again, because T is linear, we have

$$T(\vec{x} + \vec{u}) = T(\vec{x}) + T(\vec{u}) = \vec{0} + \vec{v} = \vec{v};$$

therefore, $\vec{x} + \vec{u}$ is in $T^{-1}(\vec{v})$.

So we've shown both that any element in $T^{-1}(\vec{v})$ can be written as the sum of $\vec{u}$ with an element of the null space of T, and furthermore that any such sum is an element of $T^{-1}(\vec{v})$. Therefore, these two sets are equal! $\square$

People sometimes call these $T^{-1}(\vec{v})$ sets the "fibers" of the linear map T.

A similar theorem to the above is the following:

Definition. A map $T: U \to V$ is called an **injection** if the following holds: whenever $\vec{x}, \vec{y}$ are such that $T(\vec{x}) = T(\vec{y})$, then $\vec{x} = \vec{y}$. In essence, this statement says that no two different elements in U can map to the same element in V.

Theorem 2. *A linear map $T : U \to V$ is an injection if and only if its null space consists of the single element $\vec{0}$.*

Proof. Suppose that a linear map T has two different $\vec{x} \neq \vec{y}$ such that $T(\vec{x}) = T(\vec{y})$.

Then

$$T(\vec{x} - \vec{y}) = T(\vec{x}) - T(\vec{y}) = \vec{0};$$

therefore, $\vec{x} - \vec{y}$ is in the null space of T. Moreover, this element is nonzero, because $\vec{x} \neq \vec{y}$.

Consequently, we have that whenever T is not an injection, T has some element $\vec{x} - \vec{y}$ that is in the null space that is nonzero.

This is the first half of our proof; we've shown that whenever T is not an injection, then there is something nontrivial in the null space.

To do the other half, we need to show that whenever T is an injection, then there is nothing other than 0 in the null space.

Well: on one hand, because T is linear, we know that T must map 0 to 0, because

$$T(\vec{0}) = T(\vec{0} + \vec{0}) = T(\vec{0}) + T(\vec{0}) = 2T(\vec{0}),$$

and the only vector that is equal to twice itself is the zero vector.

So we know that $\vec{0}$ is in T.

As well, because T is an injection, we know that no value in V is mapped to by more than one element. Therefore, because $\vec{0}$ is mapped to $\vec{0}$ and T is an injection, no other element can also map to $\vec{0}$; i.e. the null space of T is precisely the one element $\vec{0}$. $\square$