Math 108a	Professor: Padraic Bartlett

Lecture 11: Understanding Null Space

Week 4	*UCSB 2013*

In our last lecture, we started studying the motivation behind the concept of the **null space**. In today's talk, we return to this study.

1 Null Space: The Theorem

In our last class, we stated but did not have time to prove the following theorem:

Theorem 1. *Let $T : U \to V$ be a linear map. Let $N(T)$ denote the null space of T, and $\vec{u}, \vec{w}$ be any pair of vectors from U, V respectively such that $T(\vec{u}) = \vec{v}$.*

Let $T^{-1}(\vec{v})$ denote the set of all vectors in U that get mapped to $\vec{v}$ by T: i.e.

$$A_{\vec{v}} = \{\vec{w} \in U \mid T(\vec{w}) = \vec{v}\}.$$

Then $T^{-1}(\vec{v})$ is just $N(T)$ translated by $\vec{u}$! In other words,

$$T^{-1}(\vec{v}) = \{\vec{w} \in U \mid \textit{there is some } \vec{x} \in N(T) \textit{ such that } \vec{w} = \vec{x} + \vec{u}\}$$

In other words, understanding the collection of elements that all get mapped to $\vec{0}$ basically lets us understand the collection of elements that get mapped to any fixed vector $\vec{v}$.

We prove it here.

Proof. Let $\vec{u}, \vec{w}$ be any pair of vectors from U, V respectively such that $T(\vec{u}) = \vec{v}$.

Take any vector $\vec{w} \in T^{-1}(\vec{v})$. By definition, we know that $T(\vec{w}) = \vec{v}$.

Look at the vector $\vec{w} - \vec{u}$. If we use the fact that T is linear, we can see that

$$T(\vec{w} - \vec{u}) = T(\vec{w}) - T(\vec{u}) = \vec{v} - \vec{v} = \vec{0};$$

therefore, $\vec{w} - \vec{u}$ is in the null space $N(T)$ of T. Therefore, we can write

$$\vec{w} = (\vec{w} - \vec{u}) + \vec{u};$$

i.e. we can write $\vec{w}$ as the sum of an element from $N(T)$ and the vector $\vec{u}$.

Now, take any vector $\vec{x} \in N(T)$. Again, because T is linear, we have

$$T(\vec{x} + \vec{u}) = T(\vec{x}) + T(\vec{u}) = \vec{0} + \vec{v} = \vec{v};$$

therefore, $\vec{x} + \vec{u}$ is in $T^{-1}(\vec{v})$.

So we've shown both that any element in $T^{-1}(\vec{v})$ can be written as the sum of $\vec{u}$ with an element of the null space of T, and furthermore that any such sum is an element of $T^{-1}(\vec{v})$. Therefore, these two sets are equal! □

People sometimes call these $T^{-1}(\vec{v})$ sets the "fibers" of the linear map T.

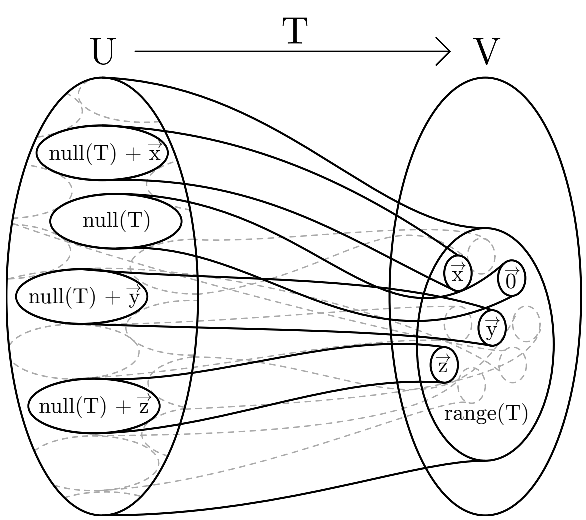
This theorem, hopefully, gives us some idea why we care about the null space: if we understand $T^{-1}(\vec{0})$, then we actually understand $T^{-1}(\vec{a})$, for **any** vector $\vec{a}$! That's powerful, and surprising.

But wait, there's more! Not only does this tell us what these $T^{-1}(\vec{a})$ things look like, it actually tells us what the entirety of U looks like in terms of the null space! Specifically, make the following two observations:

- Take any $\vec{u}$ in U. There is some set $T^{-1}(\vec{v})$ such that $\vec{u} \in T^{-1}(\vec{v})$. Specifically, just look at $T(\vec{u})$: this is equal to some element $\vec{a}$ in V. Then $\vec{u} \in T^{-1}(\vec{a})$, by definition.
- No vector $\vec{u}$ is in two different sets $T^{-1}(\vec{v}), T^{-1}(\vec{w})$. This is because if we apply T to any element in $T^{-1}(\vec{v})$, we get $\vec{v}$ by definition; similarly, if we apply T to any vector in $T^{-1}(\vec{w})$, we get $\vec{w}$ by definition. Therefore, if we had an element $\vec{u}$ in both sets, applying T to $\vec{u}$ would have to yield $\vec{v}$ and $\vec{w}$ simultaneously, which is only possible if $\vec{v} = \vec{w}$.

So the sets $T^{-1}(\vec{a})$ "partition" the set U: i.e. we can divide U up into various copies of these $T^{-1}(\vec{v})$ sets, such that every element of U is in exactly one of these sets! In other words, if we have a linear map $T : U \to V$, we can "chop up" U into a bunch of translated copies of the null space of T.

The diagram below, sketched in our last class, may help you visualize this:



To make this diagram more concrete, consider the following example:

Example. Consider the linear map $T : \mathbb{R}^2 \to \mathbb{R}$, defined by $T(x, y) = 2x - y$. What is the null space of this map? What do the sets $T^{-1}(a)$ look like, for various values of $a \in \mathbb{R}$?

Answer. The null space of this map, by definition, is the set

$$\text{null}(T) = \{(x, y) \mid T(x, y) = 0\}.$$

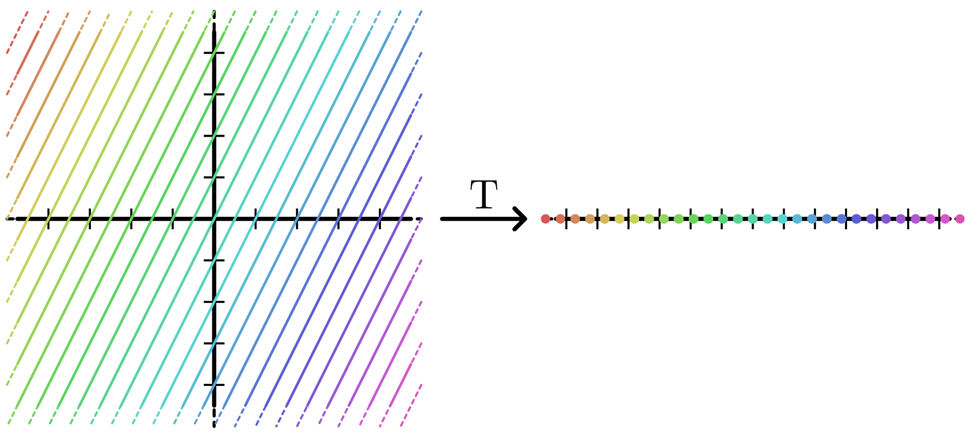
We know that $T(x, y) = 0$ if and only if $2x - y = 0$; in other words, whenever $2x = y$. Therefore, the null space of T can be more succinctly described as the set

$$\text{null}(T) = \{(x, 2x) \mid x \in \mathbb{R}\}.$$

Furthermore, notice that for any $a \in \mathbb{R}$, we have $T(a, 0) = a$. Therefore, our theorem above tells us that we can express $T^{-1}(a)$ as the null space of T shifted by $(a, 0)$: i.e.

$$T^{-1}(a) = \{(a + x, 2x) \mid x \in \mathbb{R}\}$$

Consequently, we can "partition" U into these $T^{-1}(a)$-sets, all of which are lines with slope 2 through the point $(a, 0)$; each of these sets is then mapped to their corresponding value a by T. This can be visualized by the rather beautiful picture below:



Before we started this pair of talks, we already understood why we cared about the range of a linear map T — it let us talk about the "outputs" of T. In a sense, the aim of these two talks has been to show that understanding the null space of a linear map T performs a similar task: it gives us a ton of information about the "inputs" of T.