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Lecture 12: Injection, Surjection and Linear Maps

Week 4 *UCSB 2013*

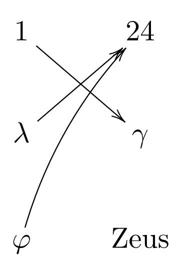
Today's lecture is centered around the ideas of **injection** and **surjection** as they relate to linear maps. While some of you may have seen these terms before in Math 8, many of you indicated in class that a quick refresher talk on the concepts would be valuable. We do this here!

1 Injection and Surjection: Definitions

Definition. A **function** f with domain A and codomain B, formally speaking, is a collection of pairs (a, b), with $a \in A$ and $b \in B$, such that there is exactly one pair (a, b) for every $a \in A$. Informally speaking, a function $f : A \to B$ is just a map which takes each element in A to an element in B.

Examples.

- $f : \mathbb{Z} \to \mathbb{N}$ given by $f(n) = 2|n| + 1$ is a function.
- $g : \mathbb{N} \to \mathbb{N}$ given by $g(n) = 2|n| + 1$ is also a function. It is in fact a different function than f, because it has a different domain!
- $j : \mathbb{N} \to \mathbb{N}$ defined by $h(n) = n^2$ is yet another function
- The function j depicted below by the three arrows is a function, with domain $\{1, \lambda, \varphi\}$ and codomain $\{24, \gamma, \text{ Zeus}\}$:



It sends the element 1 to γ, and the elements λ, φ to 24. In other words, $h(1) = \gamma$, $h(\lambda) = 24$, and $h(\varphi) = 24$.

Definition. We call a function f **injective** if it never hits the same point twice – i.e. for every $b \in B$, there is **at most one** $a \in A$ such that $f(a) = b$.

Examples. The function h from before is not injective, as it sends both λ and φ to 24:

$$\begin{tikzcd}
1 \arrow[dr] & 24 \\
\lambda \arrow[ur] & \gamma \\
\varphi \arrow[uur, bend left] & \text{Zeus}
\end{tikzcd}$$

However, suppose that we add a new element π to our codomain, and make φ map to π. Then, this modified function is now injective, because no two elements in its domain are sent to the same place:

$$\begin{tikzcd}
1 \arrow[dr] & 24 \\
\lambda \arrow[ur] & \gamma \\
\varphi \arrow[dr, bend right] & \text{Zeus} \\
 & \pi
\end{tikzcd}$$

A converse concept to the idea of injectivity is that of **surjectivity**, as defined below:

Definition. We call a function f **surjective** if it hits every single point in its codomain – i.e. if for every $b \in B$, there is **at least one** $a \in A$ such that $f(a) = b$.

Examples. The function h from before is not injective, as it doesn't send anything to Zeus:

$$\begin{tikzcd}
1 \arrow[dr] & 24 \\
\lambda \arrow[ur] & \gamma \\
\varphi \arrow[uur, bend left] & \text{Zeus}
\end{tikzcd}$$

However, if we add a new element ρ to our domain, and make ρ map to Zeus, our function is now surjective, as it hits all of the elements in its codomain:

$$\begin{tikzcd}
1 \arrow[dr] & 24 \\
\lambda \arrow[ur] & \gamma \\
\varphi \arrow[uur, bend left] & \text{Zeus} \\
\rho \arrow[ur, bend right] &
\end{tikzcd}$$

Definition. We call a function **bijective** if it is both injective and surjective.

Examples. The function

$$\begin{tikzcd}
1 \arrow[dr] & 24 \\
\lambda \arrow[ur] & \gamma \\
\varphi \arrow[dr, bend right] & \text{Zeus} \\
& \pi
\end{tikzcd}$$

from our first example is not a bijection, because it is not a surjection: nothing is mapped to the element Zeus.

Similarly, the function

$$\begin{tikzcd}
1 \arrow[dr] & 24 \\
\lambda \arrow[ur] & \gamma \\
\varphi \arrow[uur, bend left] & \text{Zeus} \\
\rho \arrow[ur, bend right]
\end{tikzcd}$$

from our second example is not a bijection, because it is not an injection: two different elements λ, φ are mapped to 24.

However, the function

$$\begin{tikzcd}
1 \arrow[dr] & 24 \\
\lambda \arrow[ur] & \gamma \\
\varphi \arrow[dr, bend left] & \text{Zeus} \\
\rho \arrow[ur, bend right] & \pi
\end{tikzcd}$$

is a bijection, as we can easily check: every element in the codomain is mapped to by some element in the domain, and no two elements in the domain are mapped to the same element.

2 Injection and Surjection: Relations to Linear Maps

Let $T : U \to V$ be a linear map between two vector space U, V. In this situation, the two concepts of **injection** and **surjection** are intimately related to the ideas of **null space** and **range** that we've been studying over the last week, in the following ways:

- A map T is a surjection if and only if $\text{range}(T) = V$. This is literally from the definition of range, as the only way that range(T) can equal V is if every element of V can arise as some output of T: i.e. if T is surjective.
- Slightly less obviously, a map T is an injection if and only if $\text{null}(T) = \vec{0}$. This comes from the theorem we proved in our last class, where we showed that whenever $T(\vec{x}) = \vec{y}$, then the sets $T^{-1}(\vec{y})$ have the form $\{\vec{x} + \vec{n} \mid \vec{n} \in \text{null}(T)\}$. If null($T$) is the set containing the single element $\{\vec{0}\}$, then the only element in $T^{-1}(\vec{y})$ is $\vec{x} + \vec{0} = \vec{x}$. Therefore, given any element $\vec{y}$ in the codomain V, there is at most one element in the domain U that maps to $\vec{y}$, because there is at most one element in $T^{-1}(\vec{y})$!

This is convenient for us in a number of ways: it is often easier to just study the null space rather than try to look at every set $T^{-1}(\vec{y})$, which cuts down on a lot of the work involved in checking if something is injective!

We make one final definition in this class:

Definition. Let U, V be a pair of vector spaces. A map $T : U \to V$ is called an **isomorphism** if it is a linear map and a bijection.

Our discussion above gives us the following theorem about isomorphisms:

Theorem. Let $T : U \to V$ be a linear map between two vector space U, V. T is an isomorphism if and only if the following two conditions hold:

- $\text{range}(T) = V$.
- $\text{null}(T) = \{\vec{0}\}$.

Essentially, two vector spaces are called **isomorphic** if they are in some senses "the same," up to a relabeling. To be more specific: consider the map $T : U \to V$ as a way to "relabel" the elements of U as elements of V. Because T is a bijection, every element of U is sent to some element in V, and no element in V is mapped to by two different elements in U. Consequently, you can think of this map T as just a way of "assigning" the elements of U to all of the elements of V. Furthermore, this map is linear, so it "plays nicely with addition and multiplication" – i.e. if we add and multiply elements before or after our relabeling, it doesn't matter! Therefore, this map T gives us a way to look at our elements of U in a way such that

- these elements of U are all matched up with elements of V, and
- the operations $+, \cdot$ on U are equivalent to the operations $+, \cdot$ on V.

Therefore, in a sense, these two spaces are basically "the same" as vector spaces.

To make this concrete, consider the following example:

Examples. The map $T : \mathbb{R}^2 \to \mathcal{P}_1(\mathbb{R})$, defined by

$$T(a, b) = a + bx$$

is an isomorphism.

Proof. We first notice that this map is linear:

- For any $(a, b), (c, d)$, we have

$$\begin{aligned} T((a, b) + (c, d)) &= T(a + c, b + d) = a + c + (b + d)x \\ T(a, b) + T(c, d) &= a + bx + c + dx = (a + c) + (b + d)x. \end{aligned}$$

  Therefore this map is additive.

- For any $(a, b), \lambda$, we have

$$\begin{aligned} T(\lambda(a, b)) &= T(\lambda a + \lambda b) = \lambda a + \lambda bx \\ \lambda T(a, b) &= \lambda(a + bx) = \lambda a + \lambda bx. \end{aligned}$$

  Therefore this map is homogenous.

This proves our map is linear.

To check surjectivity, we calculate the range:

$$\begin{aligned} \text{range}(T) &= \{T(a, b) \mid (a, b) \in \mathbb{R}^2\} \\ &= \{a + bx \mid (a, b) \in \mathbb{R}^2\} \\ &= \mathcal{P}_1(\mathbb{R}), \text{ the codomain.} \end{aligned}$$

Therefore this map is surjective.

Finally, to check injectivity, we calculate the null space:

$$\begin{aligned} \text{null}(T) &= \{(a, b) \in \mathbb{R}^2 \mid T(a, b) = 0\} \\ &= \{(a, b) \in \mathbb{R}^2 \mid a + bx = 0\} \\ &= \{(a, b) \in \mathbb{R}^2 \mid a = 0, b = 0\} \\ &= \{(0, 0)\}. \end{aligned}$$

Therefore this map is injective. We have checked all of the properties required by a map to be an isomorphism; therefore, this map is an isomorphism. $\square$

It's worth noting that these two isomorphic spaces do agree with our earlier claim that "isomorphic things are pretty much the same as vector spaces." $\mathcal{P}_1(\mathbb{R})$, the set of all polynomials of degree at most 1 with real-valued coefficients, might seem pretty different to $\mathbb{R}^2$ on a first glance. However, whenever we've worked with it as a vector space, there's really been no difference between the things we can make with $a + bx$'s and ordered pairs (a, b): they're basically two different ways of writing down the same information!

This is part of the idea behind isomorphism: it's a way of letting us know when two things are "basically the same" as vector spaces. This is one of the fundamental ideas in mathematics in linear algebra; often, we will be presented with some terrible/awful-looking thing, and have to prove that it has certain properties. The language and concept of isomorphism is a technique we have to get rid of some of that awfulness — we can often turn something that looks difficult to work with into a simpler-looking object that's "essentially the same" with an isomorphism!

You can already see this idea at work with the example we have above: many of you are comfortable with $\mathbb{R}^2$ as a vector space, but get uneasy when presented with the idea of polynomials as vectors. This isomorphism is a way of transforming the space you're comfortable in into one that you may be less comfortable with: by using it, you can just work in the simpler space $\mathbb{R}^2$, and simply use the isomorphism as a way of translating results about $\mathbb{R}^2$ into ones about the space $\mathcal{P}_1(\mathbb{R})$!