Lecture 13: Injection and Surjection: Further Examples

Our last class introduced the ideas of **injection** and **surjection**, and mentioned how these objects connected to linear maps. In this talk, we look at some additional examples for how these two objects work.

1 Injection and Surjection: Review

We quickly remind the reader of the definitions we're working with:

Definition. We call a function f **injective** if it never hits the same point twice – i.e. for every $b \in B$, there is **at most one** $a \in A$ such that $f(a) = b$.

Definition. We call a function f **surjective** if it hits every single point in its codomain – i.e. if for every $b \in B$, there is **at least one** $a \in A$ such that $f(a) = b$.

In class on Friday, we proved the following results

- A map T is a surjection if and only if $\text{range}(T) = V$.
- A map T is an injection if and only if $\text{null}(T) = \vec{0}$.

We also made the following definition:

Definition. Let U, V be a pair of vector spaces. A map $T : U \to V$ is called an **isomorphism** if it is a linear map and a bijection.

We looked at some examples of these definitions and concepts, but didn't really get enough time in our last class to properly understand the ideas here. To fix this, we devote this talk to working some additional examples.

2 Injection and Surjection: Review

Question. Let T be a linear map from $\mathbb{R}^2 \to \mathbb{R}^3$, such that $\text{range}(T) = \{(x, y, x+y) \mid x, y \in \mathbb{R}\}$. Show that T must be injective.

Proof. We proceed by contradiction, because that's always a good starting point when you don't know where to start. Suppose not: that T is not injective. Therefore, there must be some pair of points $(c, d) \neq (e, f)$ such that $T(c, d) = T(e, f)$. Using linearity, this gives us that $T(c, d) - T(e, f) = (0, 0, 0) = T(c - e, d - f)$; i.e. there is some nonzero point that gets mapped to $(0, 0, 0)$! Call it (α, β).

Now, make the following observations:

1. Take some vector (a, b) that is not in the span of the vector (α, β). The span of $\{(\alpha, \beta), (a, b)\}$ is geometrically a plane in $\mathbb{R}^2$: i.e. it's all of $\mathbb{R}^2$ itself!
2. Therefore, any vector in $\mathbb{R}^2$ can be written in the form $x(\alpha, \beta) + y(a, b)$.
3. Therefore, we can write

   $$\text{range}(T) = \{T(x(\alpha, \beta) + y(a, b)) \mid x, y \in \mathbb{R}\}.$$

   Because T is linear and (α, β) is mapped to $(0, 0, 0)$, we can use linearity to notice that $T(x(\alpha, \beta) + y(a, b)) = xT(\alpha, \beta) + yT(a, b)) = yT(a, b)$, and therefore that

   $$\text{range}(T) = \{yT(a, b) \mid y \in \mathbb{R}\}.$$

   In other words, the range of T is a one-dimensional object in $\mathbb{R}^3$; in particular, a line!
4. However, we are claiming that range$(T) = \{(x, y, x + y) \mid x, y \in \mathbb{R}\}$, which is in particular the plane in $\mathbb{R}^3$ containing the three points $(0, 0, 0), (1, 0, 1), (0, 1, 1)$. Because a line is not a plane, and in particular no line through $(0, 0, 0)$ can go through both $(1, 0, 1), (0, 1, 1)$, we have a contradiction! Therefore, our original assumption must be false, and we have proven that T is an injection.

□

Question. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a injective linear map. Prove that T is an isomorphism.

Proof. If T is an injective linear map, then the only condition we need to prove T satisfies is surjectivity: i.e. that for any vector $\vec{y}$ in $\mathbb{R}^3$, there is a vector $\vec{x} \in \mathbb{R}^3$ such that $T(\vec{x}) = \vec{y}$.

We prove this as follows.

1. Take the basis $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Look at the set $T(B) = \{T(1, 0, 0), T(0, 1, 0), T(0, 0, 1)\}$. We claim that this set spans $\mathbb{R}^3$. Notice that if this is true, we are done with our proof: to map to any $\vec{y} \in \mathbb{R}^3$, we simply use this spanning property and T's linearity to write

   $$\vec{y} = aT(1, 0, 0) + bT(0, 1, 0) + cT(0, 0, 1) \quad = T\left(a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)\right)$$
   $$= T(a, b, c),$$

   which demonstrates that for any $\vec{y}$, we can find a vector (a, b, c) that maps to $\vec{y}$.

   Therefore we just need to show that this set $T(B)$ spans $\mathbb{R}^3$. In fact, we're going to prove something stronger: $T(B)$ is a basis for $\mathbb{R}^3$!
2. We start by proving that $T(B)$ is linearly independent. This is not hard: take any nontrivial linear combination of the elements in $T(B)$:

   $$\begin{aligned} &aT(1, 0, 0) + bT(0, 1, 0) + cT(0, 0, 1) \\ =&T\left(a(1, 0, 0) + b(0, 1, 0) + c(0, 1, 0)+\right) \\ =&T(a, b, c). \end{aligned}$$

If this is equal to $\vec{0}$, our map cannot be injective, because $T(0,0,0)$ maps to $(0,0,0)$ (as we proved on HW#4!) and (a,b,c) is nonzero (because the a,b,c correspond to a nontrivial linear combination) by assumption. Therefore, this linear combination is nonzero! So our set is linearly independent.

3. We now claim that $T(B)$ spans $\mathbb{R}^3$. To see why, simply notice that we have a set of three linearly independent vectors in $\mathbb{R}^3$. Geometrically, any such triple must span a three-dimensional space: therefore, because they are contained within $\mathbb{R}^3$, they specifically span all of $\mathbb{R}^3$ itself!

   Therefore, we have proven that $T(B)$ spans $\mathbb{R}^3$. By the reasoning in (1), we have consequently shown that T is surjective, as desired.

$\square$