Math 108a | Professor: Padraic Bartlett

Lecture 18: Volume

Week 9 | *UCSB 2013*

1 Orthogonality: Review

A number of lecture back, we introduced the idea of **orthogonality**:

Definition. Take two vectors $(x_1, \dots x_n), (y_1, \dots y_n) \in \mathbb{R}^n$. We say that these two vectors are **orthogonal** if their dot product is 0. Alternately, we can say that two vectors are orthogonal if the angle θ between them is $\pm\pi/2$; this is a consequence of a theorem we proved in class, where we showed

$$\vec{x} \cdot \vec{y} = ||\vec{x}|| \cdot ||\vec{y}|| \cos(\theta).$$

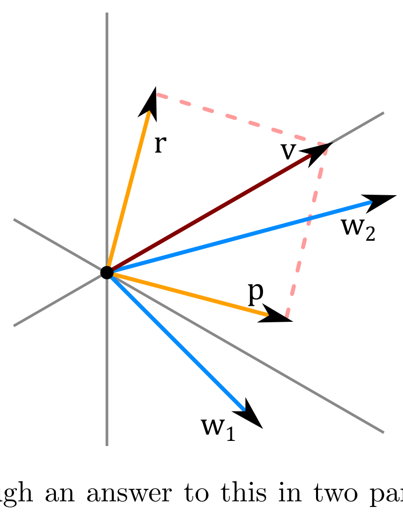
(Recall that $||\vec{x}||$ is the **length** of the vector $\vec{x}$: i.e. the length of $(1, 2, 3)$ is simply the quantity $\sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$.)

2 A Motivating Question

In this next set of lectures, we're studying the following question:

Question 1. *Suppose that we have a collection of vectors $W = \{\vec{w_1}, \dots \vec{w_k}\}$, and some other vector $\vec{v}$. Is there some way we can write $\vec{v}$ as the sum of two vectors $\vec{r} + \vec{p}$, where $\vec{r}$ is orthogonal to all of the vectors in W, while $\vec{p}$ is contained in the span of W?*

We can visualize this with the following picture. Here, we describe the red vector $\vec{v}$ as the sum of two gold vectors, one of which is orthogonal to $\vec{w_1}$ and $\vec{w_2}$, and the other of which is a linear combination of $\vec{w_1}$ and $\vec{w_2}$.

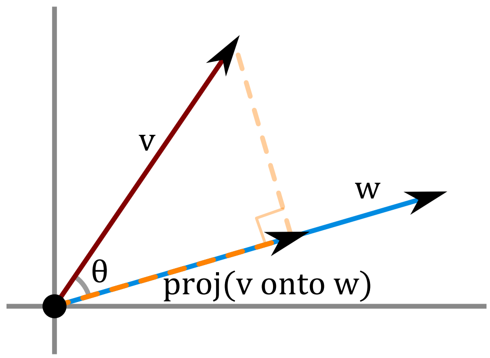


This talk takes you through an answer to this in two parts. First, make the following definition:

Definition. Let $\vec{v}, \vec{w}$ be a pair of vectors in $\mathbb{R}^n$. The **projection** of $\vec{v}$ onto $\vec{w}$, denoted proj($\vec{v}$ onto $\vec{w}$), is the following vector:

- Take the vector $\vec{w}$.
- Draw a line perpindicular to the vector $\vec{w}$, that goes through the point $\vec{v}$ and intersects the line spanned by the vector $\vec{w}$.
- proj($\vec{v}$ onto $\vec{w}$) is precisely the point at which this perpindicular line intersects $\vec{w}$.

We illustrate this below:



In particular, it bears noting that this vector is a multiple of $\vec{w}$.

A formula for this vector is the following:

$$\text{proj}(\vec{v} \text{ onto } \vec{w}) = \frac{\vec{v} \cdot \vec{w}}{||\vec{w}||^2} \cdot \vec{w}.$$

To see why, simply note that the vector we want is, by looking at the above picture, something of length $\cos(\theta) \cdot ||\vec{v}||$, in the direction of $\vec{w}$. In other words,

$$\text{proj}(\vec{v} \text{ onto } \vec{w}) = \cos(\theta) \cdot ||\vec{v}|| \cdot \frac{\vec{w}}{||\vec{w}||}.$$

Now, use the angle form of the dot product to see that because $\vec{w} \cdot \vec{v} = ||\vec{w}||||\vec{v}|| \cos(\theta)$, we have

$$\text{proj}(\vec{v} \text{ onto } \vec{w}) = \frac{\vec{v} \cdot \vec{w}}{||\vec{v}||||\vec{w}||}||\vec{v}||\frac{\vec{w}}{||\vec{w}||}.$$

Canceling the $||\vec{v}||$'s gives us the desired formula.

Using this, you can define the "orthogonal part" of $\vec{v}$ over $\vec{w}$ in a similar fashion:

Definition. Let $\vec{v}, \vec{w}$ be a pair of vectors in $\mathbb{R}^n$. The **orthogonal part** of $\vec{v}$ over $\vec{w}$, denoted orth($\vec{v}$ onto $\vec{w}$), is the following vector:

$$\text{orth}(\vec{v} \text{ onto } \vec{w}) = \vec{v} - \text{proj}(\vec{v} \text{ onto } \vec{w})$$

It bears noting that this vector lives up to its name, and is in fact orthogonal to $\vec{w}$. This is not hard to see: just take the dot product of $\vec{w}$ with it! This yields

$$\begin{aligned}
\vec{w} \cdot (\vec{v} - \text{proj}(\vec{v} \text{ onto } \vec{w})) &= \vec{w} \cdot \vec{v} - \vec{w} \cdot \text{proj}(\vec{v} \text{ onto } \vec{w}) \\
&= \vec{w} \cdot \vec{v} - \vec{w} \cdot (\frac{\vec{v} \cdot \vec{w}}{||\vec{w}||^2} \cdot \vec{w}) \\
&= \vec{w} \cdot \vec{v} - \frac{\vec{v} \cdot \vec{w}}{||\vec{w}||^2}(\vec{w} \cdot \vec{w}) \\
&= \vec{w} \cdot \vec{v} - \frac{\vec{v} \cdot \vec{w}}{||\vec{w}||^2}(w_1^2 + \ldots w_n^2) \\
&= \vec{w} \cdot \vec{v} - \frac{\vec{v} \cdot \vec{w}}{||\vec{w}||^2}(||\vec{w}||^2) \\
&= 0.
\end{aligned}$$

Therefore, in the case where $\{\vec{w_1}, \ldots \vec{w_n}\}$ is a set containing just one vector $\vec{w}$, we've answered our problem! We can write

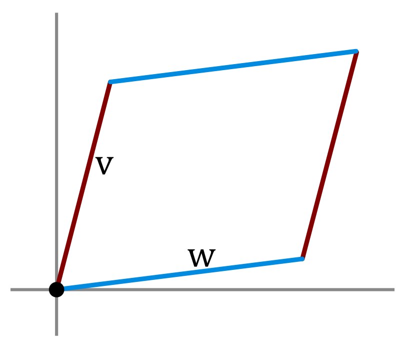
$$\vec{v} = \text{proj}(\vec{v} \text{ onto } \vec{w}) + \text{orth}(\vec{v} \text{ onto } \vec{w}),$$

where proj is a multiple of $\vec{w}$ and orth is orthogonal to $\vec{w}$!

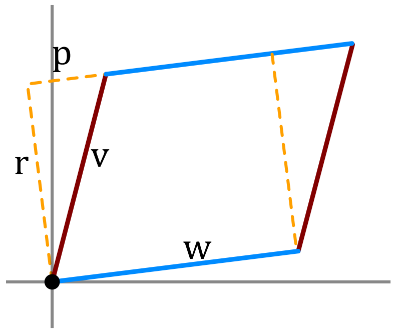
3 Orthogonality: Why Do We Care?

Orthogonality is a concept we talked about a long time ago; why return to it now?

The rough idea for why we care about orthogonality now is because it's the easiest way to understand the idea of **n-dimensional volume**! Specifically: suppose you have a parallelogram spanned by the two vectors $\vec{v}, \vec{w}$.

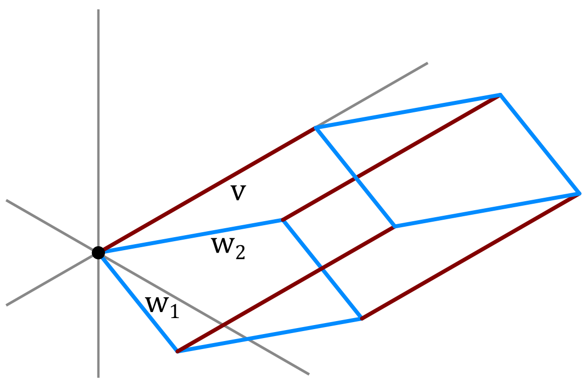


What's the area of this parallelogram? Well, it's the length of the **base** times the **height**, if you remember your high-school geometry! But what are these two quantities? Well: the **base** has length just given by the length of $\vec{w}$. The **height**, however, is precisely the kind of thing we've been calculating in this set! Specifically: suppose that we can write $\vec{v}$ as the sum $\vec{p} + \vec{r}$, where $\vec{p}$ is some multiple of $\vec{v}$ and $\vec{r}$ is orthogonal to $\vec{w}$. Then the length of $\vec{r}$ is precisely the **height**!

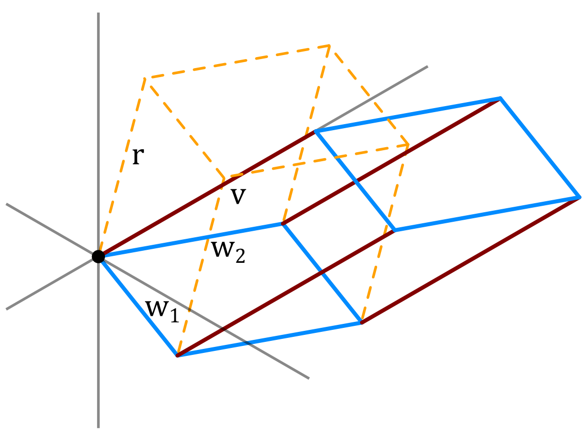


Therefore, to find the area here, we just need to multiply the length of $\vec{r}$ and the length of $\vec{w}$ together.

For three dimensions, the picture is similar. Suppose you want to find the volume of a parallelepiped — i.e. the three-dimensional analogue of a parallelogram – spanned by the three vectors $\vec{v}, \vec{w_1}, \vec{w_2}$.



What's the volume of this parallelotope? Well, this is not much harder to understand than the two-dimensional case: it's just the **area** of the parallelogram spanned by the two vectors $\vec{w_1}, \vec{w_2}$ times the **height**! In other words, suppose that we can write $\vec{v} = \vec{r} + \vec{p}$, for some vector $\vec{p}$ in the span of $\vec{w_1}, \vec{w_2}$ and some vector $\vec{r}$ orthogonal to $\vec{w_1}, \vec{w_2}$. Then the length of this vector $\vec{r}$ is, again, precisely the **height**!



This process generalizes to n dimensions: to find the volume of a n-dimensional parallelotope spanned by n vectors $\vec{w_1}, \dots \vec{w_n}$, we just start with $\vec{w_1}$, and repeatedly for each $\vec{w_2}, \vec{w_n}$, find the "height" of each $\vec{w_i}$ over the set $\vec{w_1}, \dots \vec{w_{i-1}}$ by doing this "write $\vec{w_i}$ as a projection $\vec{p}$ onto $\{\vec{w_1}, \dots \vec{w_{i-1}}\}$, plus an orthogonal bit $\vec{r}$, whose length is the height" trick. By taking the product of all of these heights, we get what we would expect to be the n-dimensional volume of the parallelotope! (In fact, it's kinda confusing just what n-dimensional volume even **means**, so if you want you can take this as the **definition** of volume for these kinds of objects in n-dimensional space.)

The rest of these talks will be devoted to this generalization.

4 Generalizing The "Orth" Map

Consider the following process (called the Gram-Schmidt process, formally):

- $\vec{u_1} = \vec{w_1}$.
- $\vec{u_2} = \vec{w_2} - \text{proj}(\vec{w_2} \text{ onto } \vec{u_1})$.
- $\vec{u_3} = \vec{w_3} - \text{proj}(\vec{w_3} \text{ onto } \vec{u_1}) - \text{proj}(\vec{w_3} \text{ onto } \vec{u_2})$.
- $\vec{u_4} = \vec{w_4} - \text{proj}(\vec{w_4} \text{ onto } \vec{u_1}) - \text{proj}(\vec{w_4} \text{ onto } \vec{u_2}) - \text{proj}(\vec{w_4} \text{ onto } \vec{u_3})$.

$\vdots$

- $\vec{u_n} = \vec{w_n} - \text{proj}(\vec{w_n} \text{ onto } \vec{u_1}) - \dots - \text{proj}(\vec{w_n} \text{ onto } \vec{u_{n-1}})$.

Fun facts: this set is made out of vectors that are all pairwise orthogonal, and it has the same span as $\{\vec{w_1}, \dots \vec{w_n}\}$.

We prove this here! The proof is kinda tricky, so if the following doesn't make perfect sense, don't worry about it — what you need to be able to do here is really just **use** the result above.

Proof. 1. The set $\{\vec{u_1}, \dots \vec{u_n}\}$ has the same span as the set $\{\vec{w_1}, \dots \vec{w_n}\}$.

To see this, simply notice that for any $\vec{w_k}$, we have

$$\vec{w_k} = \vec{u_k} + \text{proj}(\vec{w_k} \text{ onto } \vec{u_1}) + \dots + \text{proj}(\vec{w_k} \text{ onto } \vec{u_{k-1}}).$$

Because all of the $\text{proj}(\vec{w_k} \text{ onto } \vec{u_i})$ terms are multiples of $\vec{u_i}$ by construction, this is a linear combination of $\vec{u}$'s that yields $\vec{w_i}$. This means that all of the $\vec{w_i}$'s are in the span of $\{\vec{u_1}, \dots \vec{u_n}\}$.

Conversely, note that for any $\vec{u_i}$, we are in one of the following cases:

(a) $\vec{u_1} = \vec{w_1}$. In this case, $\vec{u_1}$ is clearly in the span of $\{\vec{w_1}, \dots \vec{w_n}\}$.

(b) $\vec{u_2} = \vec{w_2} - \text{proj}(\vec{w_k} \text{ onto } \vec{u_1})$. In this case, we know that $\vec{w_2}$ is trivially in the span of $\{\vec{w_1}, \dots \vec{w_n}\}$. We also know that $\text{proj}(\vec{w_k} \text{ onto } \vec{u_1})$ is a multiple of $\vec{u_1}$, which by (a) is also in the span of $\{\vec{w_1}, \dots \vec{w_n}\}$. Therefore their combination, $\vec{u_2}$, is in the span of $\{\vec{w_1}, \dots \vec{w_n}\}$.

(c) $\vec{u_3} = \vec{w_3} - \text{proj}(\vec{w_3} \text{ onto } \vec{u_1}) - \text{proj}(\vec{w_3} \text{ onto } \vec{u_2})$. Similarly, by (a) and (b), we know that the right hand side is made out of things in the span of $\{\vec{w_1}, \ldots \vec{w_n}\}$; therefore, $\vec{u_3}$ is also in the span of $\{\vec{w_1}, \ldots \vec{w_n}\}$.

$\vdots$

(z) In general, suppose that $\vec{u_1}, \ldots \vec{u_{k-1}}$ are all in the span of $\{\vec{w_1}, \ldots \vec{w_n}\}$. Then $\vec{u_k} = \vec{w_k} - \text{proj}(\vec{w_k} \text{ onto } \vec{u_1}) - \ldots \text{proj}(\vec{w_k} \text{ onto } \vec{u_{k-1}})$ is in the span of $\{\vec{w_1}, \ldots \vec{w_n}\}$; this is because the right-hand side is made out of multiples of things in the span of $\{\vec{w_1}, \ldots \vec{w_n}\}$!

Therefore, by recursion, we have that the span of $\{\vec{w_1}, \ldots \vec{w_n}\}$! contains all of $\{\vec{u_1}, \ldots \vec{u_n}\}$.

Therefore these two sets have the same span.

2. All of the vectors $\vec{u_1}, \ldots \vec{u_n}$ are orthogonal. To see this, we again use a recursive argument:

(a) We start by noticing that because $\vec{u_2} = \vec{w_2} - \text{proj}(\vec{w_2} \text{ onto } \vec{u_1})$, vector $\vec{u_2}$ is orthogonal to $\vec{u_1}$ by construction! In particular, $\vec{u_2}$ is $\text{orth}(\vec{w_2} \text{ onto } \vec{u_1})$.

(b) Now, take $\vec{u_3}$. We claim that this, too, is orthogonal to $\vec{u_1}$! To see why, simply take the dot product of $\vec{u_1}$ and $\vec{u_3}$:

$$\begin{aligned}\vec{u_1} \cdot \vec{u_3} &= \vec{u_1} \cdot (\vec{w_3} - \text{proj}(\vec{w_3} \text{ onto } \vec{u_1}) - \text{proj}(\vec{w_3} \text{ onto } \vec{u_2})) \\ &= \vec{u_1} \cdot (\vec{w_3} - \text{proj}(\vec{w_3} \text{ onto } \vec{u_1})) + \vec{u_1} \cdot (\text{proj}(\vec{w_3} \text{ onto } \vec{u_2})) .\end{aligned}$$

Again, the first dot product is just $\vec{u_1} \cdot \text{orth}(\vec{w_3} \text{ onto } \vec{u_1})$, and is therefore 0. Moreover, by (a), we know that $\vec{u_1}$ and $\vec{u_2}$ are orthogonal; therefore, the second dot product is also 0. So the sum is 0!

$\vdots$

(z) In general, if we know that $\vec{u_1}, \ldots \vec{u_{k-1}}$ are all orthogonal, we can see that $\vec{u_1} \cdot \vec{u_k}$ is zero by just calculating

$$\begin{aligned}\vec{u_1} \cdot \vec{u_k} &= \vec{u_1} \cdot (\vec{w_k} - \text{proj}(\vec{w_k} \text{ onto } \vec{u_1}) - \ldots \text{proj}(\vec{w_k} \text{ onto } \vec{u_{k-1}})) \\ &= \vec{u_1} \cdot (\vec{w_k} - \text{proj}(\vec{w_k} \text{ onto } \vec{u_1})) + \vec{u_1} \cdot (\text{proj}(\vec{w_k} \text{ onto } \vec{u_2})) + \ldots + \vec{u_1} \cdot (\text{proj}(\vec{w_k} \text{ onto } \vec{u_{k-1}}))\end{aligned}$$

Therefore, by recursion, we have that $\vec{u_1}$ is orthogonal to all of the $\vec{u_i}$'s!

We can show that $\vec{u_2}$ is orthogonal to all of the other vectors in the same way.

First, we note that we've already shown that $\vec{u_1}$ and $\vec{u_2}$ are orthogonal, so that's our first step.

Then, we can take $\vec{u_3}$, and perform the same trick as before, where we write

$$\begin{aligned}\vec{u_2} \cdot \vec{u_3} &= \vec{u_2} \cdot (\vec{w_3} - \text{proj}(\vec{w_3} \text{ onto } \vec{u_1}) - \text{proj}(\vec{w_3} \text{ onto } \vec{u_2})) \\ &= \vec{u_2} \cdot (\vec{w_3} - \text{proj}(\vec{w_3} \text{ onto } \vec{u_2})) + \vec{u_2} \cdot (\text{proj}(\vec{w_3} \text{ onto } \vec{u_1})) .\end{aligned}$$

Again, we've separated our $\vec{u_3}$ into something of the form orth($\vec{w_3}$ onto $\vec{u_2}$) and something orthogonal to $\vec{u_2}$ by a previous step (specifically, $\vec{u_1}$ and $\vec{u_2}$ being orthogonal).

In general, we can just do the same thing as before: going in order from $k = 3$ on up, for each $\vec{u_k}$, we break the dot product of $\vec{u_2} \cdot \vec{u_k}$ into a number of different dot products. Each of these dot products will be zero by previous work, so that fixes things for us.

Now: do it all again with $\vec{u_3}$! And again, and again; this process keeps repeating, with the same proof structure. Repeating this will give you that all of these $\vec{u_i}$'s are orthogonal, as claimed!

□

This proof is a bit of a pain; we provide an example of how it runs here.

Example. Run the above process on the set $\{(1,1,0),(1,0,1),(0,1,1)\}$.

Proof. So: via the algorithm above, we define the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ as follows:

$$\begin{aligned}
\mathbf{u}_1 &= (1,1,0),\\
\mathbf{u}_2 &= (1,0,1) - \text{proj}((1,0,1) \text{ onto } (1,1,0))\\
&= (1,0,1) - \frac{(1,1,0)\cdot(1,0,1)}{(1,1,0)\cdot(1,1,0)}(1,1,0)\\
&= (1,0,1) - \frac{1}{2}(1,1,0)\\
&= \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)\\
\mathbf{u}_3 &= (0,1,1) - \text{proj}((0,1,1) \text{ onto } (1,1,0)) - \text{proj}\left((0,1,1) \text{ onto } \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)\right)\\
&= (0,1,1) - \frac{(1,1,0)\cdot(0,1,1)}{(1,1,0)\cdot(1,1,0)}(1,1,0) - \frac{\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)\cdot(0,1,1)}{\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)\cdot\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)}\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)\\
&= (0,1,1) - \frac{1}{2}(1,1,0) - 0\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)\\
&= (0,1,1) - \left(\frac{1}{2}, \frac{1}{2}, 0\right)\\
&= \left(-\frac{1}{2}, \frac{1}{2}, 1\right).
\end{aligned}$$

Fun fact: these are all now pairwise orthogonal! Our theorem tells us this directly, but we can double-check it to illustrate the idea:

$$(1,1,0)\cdot\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)=-\frac{1}{2}+\frac{1}{2}=0.$$
$$(1,1,0)\cdot\left(-\frac{1}{2},\frac{1}{2},1\right)=-\frac{1}{2}+\frac{1}{2}=0.$$
$$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)\cdot\left(-\frac{1}{2},\frac{1}{2},1\right)=-\frac{1}{4}-\frac{1}{4}+\frac{1}{2}=0.$$

□

5 Why We Care About That Theorem: A General Answer to Our Question

Why do we care about this theorem? Well: it answers our question!

Specifically: take a vector $\vec{v}$ along with a set $\{\vec{w_1},\ldots\vec{w_n}\}$. Use the process above to create a set $\{\vec{u_1},\ldots\vec{u_n}\}$ with the same span as the $\vec{w_i}$'s. Then, from the perspective of breaking $\vec{v}$ into a part in the span of $\{\vec{w_1},\ldots\vec{w_n}\}$ and a part orthogonal to all of those $\vec{w_i}$'s, the $\vec{u_i}$'s and the $\vec{w_i}$'s are interchangable! In either case, we want some part that lies in their span (which is the same for both sets) and some part that's orthogonal to everything in their span (which is the same requirement for both sets!)

However, it's far far easier to construct this object for the set $\{\vec{u_1},\ldots\vec{u_n}\}$! We just have to do the following:

$$\text{proj}(\vec{v}\text{ onto }\vec{u_1},\ldots\vec{u_n})=\text{proj}(\vec{v}\text{ onto }\vec{u_1})+\ldots+\text{proj}(\vec{v}\text{ onto }\vec{u_n}).$$

This vector is clearly in the span of the vectors $\{\vec{u_1},\ldots\vec{u_n}\}$, because it's a linear combination of those vectors!

Now, set

$$\text{orth}(\vec{v}\text{ onto }\vec{u_1},\ldots\vec{u_n})=\vec{v}-\text{proj}(\vec{v}\text{ onto }\vec{u_1},\ldots\vec{u_n}).$$

We clearly have $\vec{v}$ =proj + orth. As well, for any $\vec{u_i}$, we have that

$$\vec{u_1}\cdot\text{orth}(\vec{v}\text{ onto }\vec{u_1},\ldots\vec{u_n})=\vec{u_1}\cdot\left(\vec{v}-\text{proj}(\vec{v}\text{ onto }\vec{u_1})-\ldots-\text{proj}(\vec{v}\text{ onto }\vec{u_n})\right),$$

which is equal to

$$\vec{u_i}\cdot(\vec{v}-\text{proj}(\vec{v}\text{ onto }\vec{u_i}))+\vec{u_i}\cdot\left(-\overbrace{\text{proj}(\vec{v}\text{ onto }\vec{u_1})-\ldots-\text{proj}(\vec{v}\text{ onto }\vec{u_n})}^{\text{all of the projections except for }\vec{u_i}'s}\right).$$

The left dot product is 0, again because it's just $\vec{u_i} \cdot \text{orth}(\vec{v} \text{ onto } \vec{u_i})$. The right dot product is 0, because it's the dot product of $\vec{u_i}$ with multiples of the $\vec{u_j}$'s, for $j \neq i$, and these are all vectors that are orthogonal to each toerh!

Therefore this "orth" vector is truly orthogonal to all of the $\vec{v}$'s! This gives us exactly what we wanted at the start of this set.

We study an example, to illustrate the idea:

Example. Run the above process on the set $\vec{v} = (1,1,0)$, $\vec{w_1} = \{(1,1,1), \vec{w_2} = (2,0,0)\}$.

Proof. So: via the algorithm from before, we define the vectors $\mathbf{u}_1, \mathbf{u}_2$ as follows:

$$\begin{aligned}
\mathbf{u}_1 &= (1,1,1),\\
\mathbf{u}_2 &= (2,0,0) - \text{proj}((2,0,0) \text{ onto } (1,1,1))\\
&= (2,0,0) - \frac{(2,0,0)\cdot(1,1,1)}{(1,1,1)\cdot(1,1,1)}(1,1,1)\\
&= (2,0,0) - \frac{2}{3}(1,1,1)\\
&= \left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right).
\end{aligned}$$

Then, we can define

$$\begin{aligned}
\text{proj}(\vec{v} \text{ onto } \vec{u_1}, \vec{u_2}) &= \text{proj}((1,1,0) \text{ onto } (1,1,1)) + \text{proj}\left((1,1,0) \text{ onto } \left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right)\right)\\
&= \frac{(1,1,0)\cdot(1,1,1)}{(1,1,1)\cdot(1,1,1)}(1,1,1) + \frac{(1,1,0)\cdot\left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right)}{\left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right)\cdot\left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right)}\left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right)\\
&= \frac{2}{3}(1,1,1) + \frac{2/3}{24/9}\left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right)\\
&= \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) + \frac{1}{4}\left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right)\\
&= \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) + \left(\frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}\right)\\
&= \left(1, \frac{1}{2}, \frac{1}{2}\right),
\end{aligned}$$

and

$$\begin{aligned}
\text{orth}(\vec{v} \text{ onto } \vec{u_1}, \vec{u_2}) &= (1,1,0) - \left(1, \frac{1}{2}, \frac{1}{2}\right)\\
&= \left(0, \frac{1}{2}, -\frac{1}{2}\right).
\end{aligned}$$

Notice that we indeed have

$$\vec{v} = \left(1, \frac{1}{2}, \frac{1}{2}\right) + \left(0, \frac{1}{2}, -\frac{1}{2}\right),$$

as well as having the projection part equal to a linear combination of the two $\vec{w_1}, \vec{w_2}$ vectors

$$\frac{1}{4}(2,0,0) + \frac{1}{2}(1,1,1) = \left(1, \frac{1}{2}, \frac{1}{2}\right),$$

and having the orthogonal part orthogonal to both $\vec{w_1}, \vec{w_2}$:

$$\left(0, \frac{1}{2}, -\frac{1}{2}\right) \cdot (2,0,0) = 0$$
$$\left(0, \frac{1}{2}, -\frac{1}{2}\right) \cdot (1,1,1) = 0.$$

So it WORKS! □

6 Actually Calculating Volume

By using our results from earlier, we can calculate the volume of shapes in multiple dimensions! A quick example:

Example. Calculate the volume of the shape spanned by the three vectors $(1,1,0), (1,1,1), (2,0,0)$.

Answer. We do this as follows. First, we note that the length of the vector $(2,0,0)$ is just 2.

Then, we note that the "height" of the vector $(1,1,1)$ over the vector $(2,0,0)$ is just the length of

$$\begin{aligned}\text{orth}((1,1,1) \text{ onto } (2,0,0)) &= (1,1,1) - \text{proj}((1,1,1) \text{ onto } (2,0,0)) \\ &= (1,1,1) - \frac{(1,1,1)\cdot(2,0,0)}{(2,0,0)\cdot(2,0,0)}(2,0,0) \\ &= (1,1,1) - \frac{2}{4}(2,0,0) \\ &= (0,1,1),\end{aligned}$$

which is just $\sqrt{2}$.

Finally, we need the height of the vector $(1,1,0)$ over the base spanned by $(2,0,0), (1,1,1)$. We calculated this in our example earlier: it's the length of

$$\begin{aligned}\text{orth}(\vec{v} \text{ onto } \vec{u_1}, \vec{u_2}) &= (1,1,0) - \left(1, \frac{1}{2}, \frac{1}{2}\right) \\ &= \left(0, \frac{1}{2}, -\frac{1}{2}\right),\end{aligned}$$

which is just $\frac{1}{\sqrt{2}}$.

If we take the product of these three heights, we get $2 \cdot \sqrt{2} \cdot \frac{1}{\sqrt{2}} = 2$. So the volume is 2!

.

In future lectures: a far easier way to calculate this.