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| Math 108a | Professor: Padraic Bartlett |

# Lecture 4: Subspaces

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| *Week 1* | *UCSB 2013* |

In our last class, we introduced the formal definition of a **vector space**: i.e. a set $V$ along with a field $F$ such that the following properties hold:

- **Closure(+)**: $\forall \vec{v}, \vec{w} \in V$, we have $v + w \in V$.
- **Identity(+)**: $\exists \vec{0} \in V$ such that $\forall \vec{v} \in V$, $\vec{0} + \vec{v} = \vec{v}$.
- **Commutativity(+)**: $\forall \vec{v}, \vec{w} \in V, \vec{v} + \vec{w} = \vec{w} + \vec{v}$.
- **Associativity(+)**: $\forall \vec{u}, \vec{v}, \vec{w} \in V, (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.
- **Inverses(+)**: $\forall \vec{v} \in V, \exists$ some $-\vec{v} \in V$ such that $\vec{v} + (-\vec{v}) = 0$.
- **Closure(·)**: $\forall a \in F, \vec{v} \in V$, we have $a\vec{v} \in V$.
- **Identity(·)**: $\forall \vec{v} \in V$, we have $1\vec{v} = \vec{v}$.
- **Distributivity(+, ·)**: $\forall a \in F, \vec{v}, \vec{w} \in V, a\,(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$.

We studied a number of examples of vector spaces:

- $\mathbb{R}^n$.
- $\mathbb{C}^n$.
- $\mathbb{Q}^n$.
- $M_{\mathbb{R}}(n, n)$, the collection of all $n \times n$ matrices with real-valued entries.
- $\mathbb{R}[x]$, the collection of all polynomials with real-valued coefficients.

Today's lecture is going to focus on one very important class of vector spaces: **subspaces**! We define subspaces here:

## 1 Subspaces: Basics/Definitions

The idea for today's concept is the following: consider $\mathbb{R}^3$. This is a vector space pver $\mathbb{R}$, as we discussed last class: given any two vectors in $\mathbb{R}^3$, we can add them and scale them by real numbers.

Inside of $\mathbb{R}^3$, however, there are **other** vector spaces! Consider the following set:

$$S = \{(x, y, 0) \Big| x, y \in \mathbb{R}\}.$$

We claim that this subset of $\mathbb{R}^3$, along with the normal way of adding two vectors and scaling vectors in $\mathbb{R}^3$, is a vector space! We check this axiom-by-axiom here:

- **Closure(+)**: Adding any two vectors of the form $(x, y, 0)$ will yield another vector with $z$-component equal to 0.

- **Closure($\cdot$)**: Scaling any vector of the form $(x, y, 0)$ by a constant $a$ yields another vector $(ax, ay, 0)$, that still has $z$-component equal to 0.

- **Identity(+)**: The vector $(0, 0, 0)$ is contained in this set; because it is the additive identity for $\mathbb{R}^3$, it is also the additive identity for our subset $S$.

- **Commutativity(+)**, **Associativity(+)**, **Inverses(+)**, **Identity($\cdot$)** **Distributivity(+, $\cdot$)**: These are all immediate, given our earlier results! Specifically: we know that addition is commutative and associative for $\mathbb{R}^3$. Because $S$ is a subset of $\mathbb{R}^3$, all of its elements are elements of $\mathbb{R}^3$! Therefore, these properties carry over to $S$ automatically: we've already proven that they hold for $\mathbb{R}^3$, and elements of $S$ are all elements of $\mathbb{R}^3$!

  The same argument holds for 1 serving as the scalar multiplicative identity and for distributivity; these properties hold for $\mathbb{R}^3$, therefore they still hold when we look at some subset of $\mathbb{R}^3$.

  Additive inverses, at first glance, might look trickier: unlike the other properties, which were asserting that certain properties held for all elements in our set, this property asserts that certain elements **exist** in our set. This is a fundamentally different argument: where before we can argue that because every element in $\mathbb{R}^3$ has the property of playing nicely with commutativity/associativity/etc. we cannot obviously do the same thing here.

  We can, however, do something slightly tricky that works just as well! Recall the following facts:

  - Given any vector $\vec{v}$, the vector $(-1)\vec{v}$ is equal to $-\vec{v}$, the additive inverse of $\vec{v}$. We proved this fact for fields; an identical proof works for vector spaces, and is contained in your textbook (specifically, page 12 of Axler's LADW.)
  - We already argued that our set $S$ is closed under scalar multiplication.

  Therefore, given any $\vec{v} \in S$, we know that $(-1)\vec{v}$ is in our set: in other words, that $-\vec{v}$ is in our set! So we get that additive inverses exist as a consequence of our earlier work, without any need to actually look at our subset $S$ itself! That's nice.

So: we've proven that $S$ is a vector space! (In fact, it's a vector space that should remind you of $\mathbb{R}^2$: it's basically the same thing.)

We call such vector spaces that are "contained" within another vector space **subspaces**:

**Definition.** Let $V$ be a vector space over a field $F$, along with an addition operation $+$ and a scalar multiplication operation $\cdot$. Let $S$ be a subset of $V$. We call $S$ a **subspace** of $V$, over the same field $F$ with the same addition and scalar multiplication operations, if $S$ is a vector space with respect to those two operations. A **subspace**

We have the following result, which helps us determine whether a subset of a vector space is a vector space in its own right:

**Theorem 1.** *Let $V$ be a vector space over a field $F$, along with an addition operation $+$ and a scalar multiplication operation $\cdot$. Let $S$ be any subset of $V$. $S$ is a* ***subspace*** *of $V$ if it satisfies the following three properties:*

- ***Closure(+).***
- ***Closure(·).***
- ***Identity(+).***

*Proof.* Essentially, we proved this result when we studied the subset $S = \{(x, y, 0) | x, y \in \mathbb{R}\}$ earlier.

Specifically: if we want to show that $S$ is a subspace, we need to show that $S$ is a vector space with respect to the operations $+, \cdot$ it inherits from $V$. We are assuming here that we know both closure axioms and the identity(+) axiom: we just need to show that the other axioms hold.

Like in our earlier example, because $V$ is a vector space, we know that addition is commutative and associative for $V$, and also that 1 is the scalar identity and that distributivity holds in $V$. Because $S$ is a subset of $\mathbb{R}^3$, all of its elements are elements of $V$! Therefore, these properties carry over to $S$ automatically. To give an example of this logic: for any $a, b, c$ in $S$, we know that $(a + b) + c = a + (b + c)$ because this is true for all elements of $V$. Elements of $V$ are elements of $S$; therefore this property is held by elements of $S$ as well.

This gives us everything except for additive inverses. We show that those exist using the same logic as before: We can, however, do something slightly tricky that works just as well! Recall the following facts:

- Given any vector $\vec{v}$, the vector $(-1)\vec{v}$ is equal to $-\vec{v}$, the additive inverse of $\vec{v}$. Again, we proved this fact for fields, and an identical proof works for vector spaces and is contained in your textbook (specifically, page 12 of Axler's LADW.)
- We are assuming that our set $S$ is closed under scalar multiplication.

Therefore, given any $\vec{v} \in S$, we know that $(-1)\vec{v} = -\vec{v}$ is in our set: in other words, additive inverses exist.

So $S$ is a vector space if the three properties closure(+), closure(·), and identity(+) hold! □

This result is rather useful in classifying subspaces, because it reduces the number of axioms we need to study down to just three. Moreover, these three are usually some of the easier ones to check: as you've seen multiple times in this class, checking associativity and distributivity is irritating. But we don't have to do it for subspaces!

We consider a few quick examples and nonexamples of subspaces, in order to ground our intuition:

## 2 Subspaces: Examples

**Question.** Consider the vector space $\mathbb{R}^4$, together with its usual addition and scalar multiplcation. Take the following subset:

$$S = \{(a, b, c, d) : a + b + c + d = 0\}.$$

Is this subset a subspace of $\mathbb{R}^4$?

**Answer.** Yes! To see this, we just check our three axioms:

- **Closure(+)**: Take any two vectors $(a, b, c, d), (w, x, y, z)$ such that $a + b + c + d = 0$, $w + x + y + z = 0$. Look at their sum:

  $$(a, b, c, d) + (w, x, y, z) = (a + w, b + x, c + y, d + z).$$

  If we sum up the coördinates of this vector, we get

  $$a + w + b + x + c + y + d + z = (a + b + c + d) + (w + x + y + z) = 0 + 0 = 0.$$

  Therefore this vector is in our set $S$.

- **Closure(·)**: Take any vector $(a, b, c, d)$ such that $a + b + c + d = 0$. Choose any real number $x$. Then

  $$x(a, b, c, d) = (xa, xb, xc, xd).$$

  If we sum up the coördinates of this vector, we get

  $$xa + xb + xc + xd = x(a + b + c + d) = x \cdot 0 = 0.$$

  Therefore this vector is in our set $S$.

- **Identity(+)**. The vector $(0, 0, 0, 0)$ is trivially in our set $S$.

Therefore our subset is a subspace!

**Question.** Consider the vector space $\mathbb{R}^4$, together with its usual addition and scalar multiplcation. Take the following subset:

$$S = \{(a, b, c, d) : a + b + c + d = 1\}.$$

Is this subset a subspace of $\mathbb{R}^4$?

**Answer.** No!

To see why, just notice that the vector $(0, 0, 0, 0)$ is not in this set. Therefore, our set has no additive identity.

**Question.** Consider the vector space $\mathbb{R}[x]$ of all real-valued polynomials with finite degree, along with the usual operations $+, \cdot$ that we use to add two polynomials together and scale them by real numbers.

Consider the following subset of $\mathbb{R}[x]$:

$$S = \{p(x) | p(x) \text{ has degree 5 or less}\}.$$

Is this subset a subspace of $\mathbb{R}[x]$?

**Answer.** Yes! Again, we just check our three properties:

- **Closure(+)**: Take any two polynomials of degree 5 of less. If we add them together, we clearly get a polynomial of degree 5 or less, as it is impossible to make $x^6$ or higher-order terms by simply adding two polynomials. Therefore, our set is closed under addition.
- **Closure($\cdot$)**: Take any polynomial of degree 5. Multiply it by any real number. Its degree is either the same as it was (if that real number was nonzero), or we have the 0-polynomial (if that real number was 0.) In either case, our polynomial still has degree 5 or less. Therefore, our set is closed under multiplication.
- **Identity(+)**. The 0-polynomial, 0, has degree 5 or less, as it has no terms of the form $a_n x^n$ with $a_n \neq 0$ and $n \geq 6$. Therefore, it is in our set.

Therefore this is a subspace of $\mathbb{R}[x]$!

# 3 Sums of Subspaces

Something we like to do with subspaces is combine them. We define a rigorous way to do this here:

**Definition.** Let $V$ be a vector space over a field $F$, along with an addition operation $+$ and a scalar multiplication operation $\cdot$. Let $S, T$ be a pair of subspaces of $V$. We define the **sum** of these two subspaces, denoted $S + T$, as the set of all possible sums of elements of $S$ and $T$: in other words,

$$S + T = \{\vec{v} + \vec{w} | \vec{v} \in S, \vec{w} \in T\}.$$

Fun fact: the sum of any two subspaces is a subspace! This will be on your HW.

Instead, we illustrate how to take these sums with a few quick examples:

**Question.** Consider the vector space $\mathbb{R}^4$, along with the two subspaces

$$\begin{aligned} S &= \{(w, x, 0, 0) : w, x \in \mathbb{R}\}, \\ T &= \{(0, 0, y, 0) : y, z \in \mathbb{R}\}. \end{aligned}$$

What is the sum $S + T$?

**Answer.** So, by definition, the elements of $S + T$ are all of the elements in $\mathbb{R}^4$ that we can write in the form

$$(w, x, 0, 0) + (0, 0, y, 0) = (w, x, y, 0), \text{ for any } w, x, y \in \mathbb{R}.$$

In other words, $S + T$ is the collection of all vectors in $\mathbb{R}^4$ with last coördinate equal to 0.

**Question.** Consider the vector space $\mathbb{R}^3$, along with the two subspaces

$$S = \{(x, 0, 0) : x \in \mathbb{R}\},$$
$$T = \{(a, b, c) : a, b, c \in \mathbb{R}, a + b + c = 0\}.$$

What is the sum $S + T$?

**Answer.** So, by definition, the elements of $S+T$ are all of the elements in $\mathbb{R}^4$ that we can write in the form

$$(x, 0, 0) + (a, b, c) = (x + a, b, c), \text{ for any } x \in \mathbb{R}, a + b + c = 0.$$

So: what kinds of points can be written in this form? Well: if we solve $a+b+c = 0$. we get $a = -b-c$. So we can set $b, c$ to whatever we want, and we'll still have a point $(-b-c, b, c)$ that's in $T$. So, for our points $(x + a, b, c)$, we can get any value of $b, c$ that we want by thinking of our points as triples

$$(x - b - c, b, c).$$

Can we also get whatever we want for the first coördinate? Yes! For any value $d$, let $x = d + b + c$. Then, we have for any $d, b, c \in \mathbb{R}$,

$$(d + b + c, 0, 0) + (-b - c, b, c) = (d, b, c).$$

In other words, we can express any point in $\mathbb{R}^3$ as a sum of one element in $S$ and one element in $T$. Therefore, $S + T = \mathbb{R}^3$.