Math 108a	Professor: Padraic Bartlett
Lecture 7: Dot Products	
Week 3	*UCSB 2013*

This talk is designed to introduce the **dot product**. We do this below:

1 Dot Products

Given a pair of vectors in $\mathbb{R}^n$, the **dot product** operation is the following map:

Definition. Take two vectors $(x_1, \ldots x_n), (y_1, \ldots y_n) \in \mathbb{R}^n$. Their **dot product** is simply the sum

$$x_1y_1 + x_2y_2 + \ldots x_ny_n.$$

Many of you have seen an alternate, geometric definition of the dot product:

Definition. Take two vectors $(x_1, \ldots x_n), (y_1, \ldots y_n) \in \mathbb{R}^n$. Their **dot product** is the product

$$||\vec{x}|| \cdot ||\vec{y}|| \cos(\theta),$$

where θ is the angle between $\vec{x}$ and $\vec{y}$.

(Recall, from your previous classes, that given a vector $\vec{x}$, the quantity $||\vec{x}||$ simply denotes the length of this vector. For example, if $\vec{x}$ is a vector in $\mathbb{R}^3$ of the form (a, b, c), this length is simply the quantity $\sqrt{a^2 + b^2 + c^2}$, which you can see by using the Pythagorean theorem.)

In this talk, we show that these two definitions are equivalent.

2 Equivalence of the Geometric and Algebraic Dot Products

Theorem. Let $\vec{x} = (x_1, x_2, x_3), \vec{y} = (y_1, y_2, y_3)$ be a pair of vectors in $\mathbb{R}^3$. Then the algebraic interpretation of $\vec{x} \cdot \vec{y}$, given by

$$x_1y_1 + x_2y_2 + \ldots x_ny_n,$$

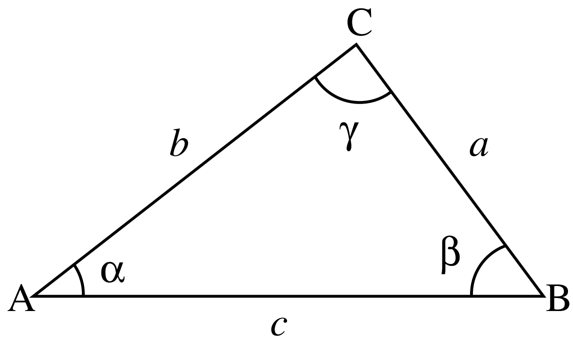
is equal to the geometric interpretation

$$||\vec{x}|| \cdot ||\vec{y}|| \cos(\theta),$$

where θ is the angle between $\vec{x}$ and $\vec{y}$.

Proof. Essentially, this is a consequence of the Law of Cosines, a trigonometry rule you may have ran into in high school. We restate it here:

Proposition. (Law of Cosines) Given the triangle

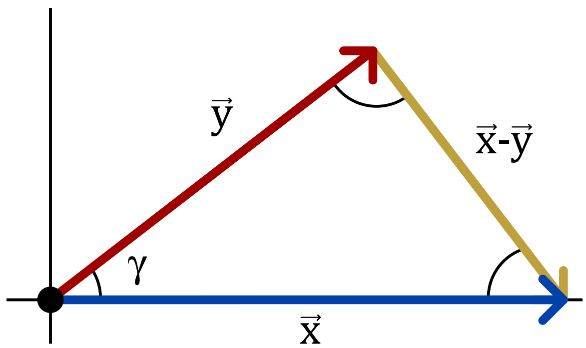


,

we have the equality

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma).$$

To see how this applies to our situation, consider the following picture:



If we apply the law of cosines to this image, we get

$$||\vec{x} - \vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2 - 2||\vec{x}||||\vec{y}||\cos(\gamma).$$

However, we know that the length of $\vec{x} - \vec{y}$ is just the length of the vector $(x_1 - y_1, x_2 - y_2, x_3 - y_3)$, which is

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y^2)^2 + (x_3 - y_3)^2}.$$

Therefore, if we square this, we get

$$\begin{aligned} ||\vec{x} - \vec{y}||^2 &= (x_1^2 - 2x_1y_1 + y_1^2) + (x_2^2 - 2x_2y_2 + y_2^2) + (x_3^2 - 2x_3y_3 + y_3^2) \\ &= (x_1^2 + x_2^2 + x_3^2) + (y_1^2 + y_2^2 + y_3^2) - 2(x_1y_2 + x_2y_2 + x_3y_3) \\ &= ||\vec{x}||^2 + ||\vec{y}||^2 - 2\vec{x}\cdot\vec{y}. \end{aligned}$$

If we plug this into our law of cosines formula, we get

$$\begin{gathered} ||\vec{x}||^2 + ||\vec{y}||^2 - 2\vec{x}\cdot\vec{y} = ||\vec{x}||^2 + ||\vec{y}||^2 - 2||\vec{x}||||\vec{y}||\cos(\gamma) \\ \Rightarrow \vec{x}\cdot\vec{y} = ||\vec{x}||||\vec{y}||\cos(\gamma). \end{gathered}$$

So we've proven our claim! □

One useful consequence of this geometric understanding of the dot product is the concept of **orthogonality**:

Definition. Two vectors $\vec{v}, \vec{w}$ are called **orthogonal** if their dot product is 0.

Geometrically, we can interpret this as saying that two vectors are orthogonal whenever the angle θ between them is such that $\cos(\theta) = 0$; i.e. that these two vectors meet at either a $90°$ or $270°$-degree angle!

3 Signals

Instead of focusing on further dot-product calculation, we close this lecture by studying a related object: **signals**, and **correlation**.

Definition. A **signal** A of length n, for the purposes of this lecture, is some finite sequence $(a_1, \ldots a_n)$ of $+1$ and -1's. For example, $(+1, -1, -1, -1)$ is a signal of length 4.

Definition. Given a signal A of length n, for any j between 0 and $n - 1$, we define the **autocorrelation coefficients** of this sequence to be the values

$$c(A, j) = \left| a_1 a_{1+j} + a_2 a_{2+j} + \ldots + a_{n-j} a_n \right|.$$

Observation. Notice that for any signal $A = (a_1, \ldots a_n)$, $c(A, 0) = n$. This is because $c(A, 0)$ is just

$$c(A, j) = \left| a_1 a_1 + a_2 a_2 + \ldots + a_n a_n \right|,$$

i.e. the absolute value of the dot product of A with itself. Because every entry of A is ± 1, the dot product of A with itself is just a sum of n copies of $(\pm 1)^2$: i.e. n copies of 1, which is just n.

Definition. A signal A is called a **Barker code** if for any $j \geq 1$, we have $c(A, j) \leq 1$.

The idea here is basically the following: suppose you have two devices trying to communicate wirelessly on some radio wavelength. You could have each device simply send signals of 1's and 0's to the other on that wavelength: but what if there's interference? Like, if there were occasionally bits of static that would get in the way, you'd lose vital bits of information. Similarly, if some other devices started transmitting on your wavelength, you might not be able to tell this "cross-traffic" apart from the device you're actually trying to communicate with.

A solution is to use these codes! Specifically: let A be a signal, as defined above. Every time you want to send a 1, instead send one copy of our signal A. Every time you want to send a 0, send -1 times a copy of A. On the receiving end, we simply take all incoming signals $(s_1, \ldots s_n)$, and correlate these signals with A: i.e. look at $|s_1 a_1 + \ldots s_n a_n|$.

If this correlation is large and close to $\pm n$, then it's likely a copy of our original signal, and we should interpret this as a ± 1 sent from the device we're communicating with! If it's small, however, it's likely some cross-traffic or just static in the air, and we can ignore it.

These **Barker codes** are incredibly useful; we first developed them somewhere after WWII as a way to solve exactly this problem for radar systems, and we actively use codes of length 11 in the 802.11b wireless standard!

We first give an example of a Barker code:

Examples. The code

$$(+1, +1, +1, -1, -1, +1, -1)$$

is a Barker code. To see this, we simply calculate its autocorrelation coefficients:

$$C(A,1) = \begin{array}{cccccccc} +1&+1&+1&-1&-1&+1&-1& \\ &\cdot&\cdot&\cdot&\cdot&\cdot&\cdot& \\ &+1&+1&+1&-1&-1&+1&-1 \\ \hline | &+1&+1&-1&+1&-1&-1&| \end{array} = 0$$

$$C(A,2) = \begin{array}{ccccccccc} +1&+1&+1&-1&-1&+1&-1&& \\ &&\cdot&\cdot&\cdot&\cdot&\cdot&& \\ &&+1&+1&+1&-1&-1&+1&-1 \\ \hline &|&+1&-1&-1&-1&+1&|& \end{array} = 1$$

$$C(A,3) = \begin{array}{cccccccccc} +1&+1&+1&-1&-1&+1&-1&&& \\ &&&\cdot&\cdot&\cdot&\cdot&&& \\ &&&+1&+1&+1&-1&-1&+1&-1 \\ \hline &&|&-1&-1&+1&+1&|&& \end{array} = 0$$

$$C(A,4) = \begin{array}{ccccccccccc} +1&+1&+1&-1&-1&+1&-1&&&& \\ &&&&\cdot&\cdot&\cdot&&&& \\ &&&&+1&+1&+1&-1&-1&+1&-1 \\ \hline &&&|&-1&+1&-1&|&&& \end{array} = 1$$

$$C(A,5) = \begin{array}{cccccccccccc} +1&+1&+1&-1&-1&+1&-1&&&&& \\ &&&&&\cdot&\cdot&&&&& \\ &&&&&+1&+1&+1&-1&-1&+1&-1 \\ \hline &&&&|&+1&-1&|&&&& \end{array} = 0$$

$$C(A,6) = \begin{array}{ccccccccccccc} +1&+1&+1&-1&-1&+1&-1&&&&&& \\ &&&&&&\cdot&&&&&& \\ &&&&&&+1&+1&+1&-1&-1&+1&-1 \\ \hline &&&&&|&-1&|&&&&& \end{array} = 1$$

Barker codes have been shown to exist of lengths $2, 3, 4, 5, 7, 11, 13$. Despite their utility, it is currently unknown whether Barker codes exist of other lengths.