Math 108a | Professor: Padraic Bartlett

Practice Midterm!

Wednesday, 9-10am, Phelps 3505 | *UCSB 2013*

There are three sections to this midterm:

1%: Your name. Seriously, write your name on your test.

33%: A **true-false** section. There are ten statements here. Your only task in this section is to label each statement true or false. The only thing that will be graded in this section is your answers.

66%: A **proof**-focused section. There are four problems here. Your job is to pick **two** of these problems, and solve them. Each individual problem will count towards 33% of your overall score. Solutions in this section will be graded for both their **answers** and their **arguments**. In particular, you need to **include a proof** with any problem that you solve here if you want to get a non-awful score. If you attempt more than two problems in this section, your first two answers will be graded. Choose wisely.

Resources allowed: up to **three** hand-written sheets of paper, in your own handwriting. Also, arbitrary amounts of blank paper, for scratch work. No textbooks or other notes are allowed. When proving results, you are allowed to use anything that we've proven in class or in the HW without proof. Other things, like results from LADR that we haven't proven yet or results from other books/classes, must be proven if you want to use them. It bears noting that every problem on this test can be solved using only things you have learned in this class and homework, and outside resources are borderline-useless.

All work must be written in a blue book! Work not in a blue book will be used for decorative origami and in particular not graded.

Good luck and have fun!

1 True-False

For each statement below, label it as either true or false.

1. The set $\{a + bx^2 \mid a, b \in \mathbb{R}\}$ is a subspace of $\mathcal{P}_3(\mathbb{R})$.

2. The dimension of the vector space $\mathcal{P}_2(\mathbb{R})$ is 4.

3. The null space of the map $T : \mathbb{R}^2 \to \mathbb{R}$, $T(x, y) = x$ is the set $\{(0, y) : y \in \mathbb{R}\}$.

4. The range of a linear map $T : U \to V$ is a subspace of V.

5. $\langle \mathbb{Z}/2\mathbb{Z}, +, \cdot \rangle$ is not a field.

6. The dot product of two vectors $\vec{a}, \vec{b} \in \mathbb{R}^2$ is equal to $||a|| \cdot ||b|| \cdot \sin\left(\frac{\pi}{2} - \theta\right)$, where θ is the angle between these two vectors $\vec{a}, \vec{b}$.

7. The set $(1, 1, 1, 1), (1, 1, -1, -1), (1, -1, 1 - 1), (1, -1, -1, 1)$ is linearly dependent.

8. The set of vectors $(1, 2, 3), (2, 3, 4), (3, 4, 5)$ is a basis for $\mathbb{R}^3$.

9. There is an injective linear map from $\mathbb{C}$ to $\mathbb{R}^2$.

10. Every vector space has a basis with finitely many vectors in it.

2 Proofs

1. (a) Let $L : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear map defined by $L(w, x, y, z) = (w - x, y - z)$. Let $M : \mathbb{R}^2 \to \mathbb{R}^4$ be the linear map defined by $M(x, y) = (x, y, x + y, x - y)$. Show that the composition $M \circ L : \mathbb{R}^4 \to \mathbb{R}^4$ of these two maps is not the identity[1] map.

   (b) In general, let $S : \mathbb{R}^4 \to \mathbb{R}^2, T : \mathbb{R}^2 \to \mathbb{R}^4$ be a pair of linear maps. Consider the composition $T \circ S : \mathbb{R}^4 \to \mathbb{R}^4$ of these two maps. Is it possible that $S \circ T$ is the identity map? Either find a pair of maps such that their composition is the identity, or prove no such pair can exist.

2. Let V be a vector space, and $T : V \to V$ be an injective linear map on V. Suppose that the set of vectors $\{\vec{v_1}, \ldots \vec{v_n}\}$ is a linearly independent subset of V. Prove that $\{T(\vec{v_1}), \ldots T(\vec{v_n})\}$ is a linearly independent subset of V.

3. Suppose that U, V are a pair of vector spaces and that $T : U \to V$ is an isomorphism.

   (a) Show that for any $\vec{v} \in V$, $T^{-1}(\vec{v})$ is a set containing exactly one element.

   (b) Using part (a), define the map $S : V \to U$ as follows:

   $$S(\vec{v}) = \text{ the unique vector } \vec{u} \text{ in the set } T^{-1}(\vec{v}).$$

   Prove that S is an isomorphism: i.e. show that S is a bijective linear map.

4. (a) Take an arbitrary linear map $T : \mathbb{R}^2 \to \mathbb{R}$. Prove that $\text{null}(T) \neq \{(0, 0)\}$.

   (b) Suppose that S is a linear map $\mathbb{R}^2 \to \mathbb{R}$ with $\text{null}(S) = \{(-x, 2x) : x \in \mathbb{R}\}$. Prove that S is surjective.

[1] In case you forgot: the identity map id on a vector space V is the map whose input is its output: i.e. $id(\vec{v}) = \vec{v}$, for any $\vec{v} \in V$.