Math 108a	Professor: Padraic Bartlett
Practice Final! - Topics	
Week 11	*UCSB 2013*

You have a final coming! Here is a brief listing of the main topics involved in said final, which is focused on the second half of the course. That said, questions about the second half of the course may involve you needing to know things from the first half! I could ask you to consider a matrix as a linear map, and tell me what its null space is, for example; in this situation, you'd need to know what the null space was. For an idea of what kinds of "previous knowledge" are needed, look at the practice final.

1 Final Exam: Main Topics

Chronologically, here's what we've done since the midterm:

1. **The rank-nullity theorem.** The rank-nullity is the following result:

   Theorem. Let U, V be a pair of finite-dimensional vector spaces, and let $T : U \to V$ be a linear map. Then the following equation holds:

   $$\text{dimension}(\text{null}(T)) + \text{dimension}(\text{range}(T)) = \text{dimension}(U).$$

   We used this in many situations to answer questions like

   "Is there a linear map T with domain *blah* and nullspace foo?"

   or

   "Is there an injective linear map T with domain *blah* and range $grar$?"

   In both cases, we answered our questions by

   - calculating the dimension of the two pieces of information we were given,
   - using rank-nullity to solve for the dimension of the piece of information we don't have, and
   - seeing if that number made sense.

   For example, if we asked for a linear map $T : \mathbb{R} \to \mathbb{R}^2$ with range equal to $\mathbb{R}^2$, we could easily see that no such map exists: because the dimension of the domain is 1 and the dimension of the proposed range is 2, the dimension of the proposed nullspace would have to be -1, which is nonsense (think about the definition of dimension, from lecture 6, and you'll see why it is impossible to have a "negative" dimension.)

   Similarly, if I asked you if a surjective linear map from $\mathbb{R}^2 \to \mathbb{R}^3$ could exist with nullspace given by all vectors in $\mathbb{R}^2$ with their first coordinate equal to 0, you could also note that this is impossible: the dimension of the domain is 2, while the dimension of the nullspace is 1. Therefore the dimension of the range is 1, and thus the range is in particular not all of $\mathbb{R}^3$.

Examples can be found in: Lecture 14.

2. **Matrices**! We learned a number of things about matrices:

   - How to turn linear maps into matrices! We did this by taking a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ and forming the $m \times n$ array

$$T_{\text{matrix}} = \begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ T(\vec{e_1}) & T(\vec{e_2}) & \dots & T(\vec{e_n}) \\ \vdots & \vdots & \dots & \vdots \end{bmatrix}.$$

   - How to multiply matrices! We showed that for any $m \times n$ matrix A with rows $\vec{a_{r_1}}, \dots \vec{a_{r_m}}$, and any vectors $\vec{x} \in \mathbb{R}^n$, we have

$$A(\vec{x}) = (\vec{x} \cdot \vec{a_{r_1}}, \vec{x} \cdot \vec{a_{r_2}}, \dots \vec{x} \cdot \vec{a_{r_m}}).$$

   Moreover, we showed that for any $m \times n$ matrix A and any $k \times m$ matrix B, we have that BA is the $k \times n$ matrix

$$\begin{bmatrix} \vec{b_{r_1}} \cdot \vec{a_{c_1}} & \vec{b_{r_1}} \cdot \vec{a_{c_2}} & \dots & \vec{b_{r_1}} \cdot \vec{a_{c_n}} \\ \vec{b_{r_2}} \cdot \vec{a_{c_1}} & \vec{b_{r_2}} \cdot \vec{a_{c_2}} & \dots & \vec{b_{r_2}} \cdot \vec{a_{c_n}} \\ \dots & \dots & \ddots & \dots \\ \vec{b_{r_k}} \cdot \vec{a_{c_1}} & \vec{b_{r_k}} \cdot \vec{a_{c_2}} & \dots & \vec{b_{r_k}} \cdot \vec{a_{c_n}} \end{bmatrix}.$$

Examples can be found in: Lecture 15.

3. **Elementary matrices**. There were three kinds of elementary matrices:

$$E_{\text{multiply } k \text{ by } \lambda} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \lambda & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{bmatrix}, \quad E_{\text{switch } k \text{ and } l} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix},$$

$$E_{\text{add } \lambda k \text{ to } l} = \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & \lambda & 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix}.$$

Specifically:

- $E_{\text{multiply entry k by }\lambda}$ is the $n \times n$ identity matrix, except with a λ in entry (k, k).
- $E_{\text{switch entry }k\text{ and entry }l}$ is the $n \times n$ identity matrix, except with its k-th and l-th columns swapped.
- $E_{\text{add }\lambda\text{ copies of entry }k\text{ to entry }l}$ is the $n \times n$ identity matrix, except with a λ in entry (l, k).

There were two major theorems we proved about elementary matrices, that we have used in lectures and the HW repeatedly in class:

- **Theorem.** Take any $n \times n$ matrix A, and look at the products EA, AE. We have the following cases:
  - If $E = E_{\text{multiply entry k by }\lambda}$, then EA is the matrix A with its k-th row multipled by λ. Similarly, AE is the matrix A with its k-th column multiplied by λ.
  - If $E = E_{\text{switch entry }k\text{ and entry }l}$, then EA is the matrix A with its k-th and l-th rows swapped. Similarly, AE is the matrix A with its k-th and l-th columns swapped.
  - If $E = E_{\text{add }\lambda\text{ copies of entry }k\text{ to entry }l}$, then EA is the matrix A with λ copies of its k-th row added to its l-th row. Similarly, AE is the matrix A with λ copies of its k-th column added to its l-th column.

  We used this theorem repeatedly whenever we multiplied a matrix by an elementary matrix, as it gave us a way to "quickly" calculate their product (as opposed to having to find the dot product of each row with each column, which takes a while.)
- **Theorem.** Let A be a $n \times n$ matrix. Then there is some string of elementary matrices $E_1, \dots E_k$ such that

  $$A = E_1 \cdot \ldots \cdot E_k.$$

  We came up with specific algorithms to explicitly find these decompositions. Given a 3×3 matrix, you should be able to find such a decomposition of this matrix into elementary matrices!

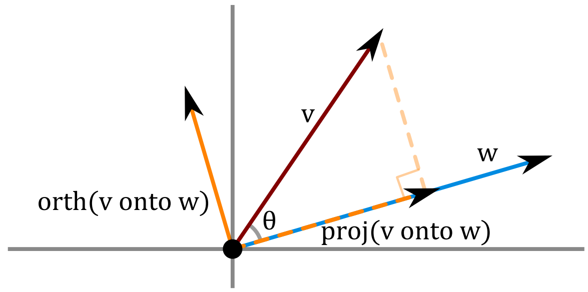
Examples can be found in: Lecture 16, Lecture 17.

4. **Volume.** We spent a week working with the concepts of **orthogonality**, **projection** and **volume**. Specifically, we defined the following concepts:

- **Definition.** Given a pair of vectors $\vec{v}, \vec{w}$ in $\mathbb{R}^n$, we define the **projection** of $\vec{v}$ onto $\vec{w}$, and the **orthogonal part** of $\vec{v}$ over $\vec{w}$, as the following objects:

  $$\begin{aligned}\text{proj}(\vec{v}\text{ onto }\vec{w}) &= \frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w},\\ \text{orth}(\vec{v}\text{ onto }\vec{w}) &= \vec{v} - \text{proj}(\vec{v}\text{ onto }\vec{w})\end{aligned}$$

  We illustrate these vectors below:



- **Definition.** Given n vectors $\{\vec{w_1}, \ldots \vec{w_n}\}$, we defined the **parallelotope** spanned by these n vectors as the set

$$\{a_1\vec{w_1} + \ldots a_n\vec{w_n} \mid 0 \leq a_i \leq 1, \forall i\}.$$

The primary use we had for these proj and orth maps was in **calculating the volume** of these parallelotopes! We showed that we could do this via the following process. First, we calculated the following vectors:

- $\vec{u_1} = \vec{w_1}$.
- $\vec{u_2} = \vec{w_2} - \text{proj}(\vec{w_2} \text{ onto } \vec{u_1})$.
- $\vec{u_3} = \vec{w_3} - \text{proj}(\vec{w_3} \text{ onto } \vec{u_1}) - \text{proj}(\vec{w_3} \text{ onto } \vec{u_2})$.
- $\vec{u_4} = \vec{w_4} - \text{proj}(\vec{w_4} \text{ onto } \vec{u_1}) - \text{proj}(\vec{w_4} \text{ onto } \vec{u_2}) - \text{proj}(\vec{w_4} \text{ onto } \vec{u_3})$.

$\vdots$

- $\vec{u_n} = \vec{w_n} - \text{proj}(\vec{w_n} \text{ onto } \vec{u_1}) - \ldots - \text{proj}(\vec{w_n} \text{ onto } \vec{u_{n-1}})$.

We thought of each of these vectors as representing the "height" of each $\vec{w_i}$ over the previous $\vec{w_1}, \ldots \vec{w_{i-1}}$. With this idea in mind, we defined the **volume** of our parallelepiped as

$$||\vec{u_1}|| \cdot ||\vec{u_2}|| \cdot \ldots \cdot ||\vec{u_n}||.$$

Examples can be found in: Lecture 18, Lecture 19.

5. **The positive determinant**. Given a $n \times n$ matrix A, we defined the **positive determinant** of A as follows:

Definition. The **positive determinant of** A, written $\det^+(A)$, is the volume of the parallelotope spanned by the columns of A.

So, in a sense, the actual calculation of these objects is something we looked at in the previous lectures! The reason we study them again here is because we proved a number of useful theorems about the positive determinant:

- **Theorem.** Take any matrix A. Look at the matrix $A \cdot E$. Then,

$$E_{\text{multiply entry } k \text{ by } \lambda}.$$

Then

– If $E = E_{\text{multiply entry k by } \lambda}$, then $\det^+(AE) = |\lambda| \cdot \det^+(A)$.
– If $E = E_{\text{switch entry } k \text{ and entry } l}$, then $\det^+(AE) = \det^+(A)$.
– If $E = E_{\text{add } \lambda \text{ copies of entry } k \text{ to entry } l}$, then $\det^+(AE) = \det^+(A)$.

We used this theorem to create a new way to calculate the positive determinant:

- **Lemma.** Take any $n \times n$ matrix A. Write A as the product of elementary matrices: i.e.

$$A = E_1 \cdot \ldots \cdot E_n.$$

Then

$$\det{}^+(A) = |\lambda_1 \cdot \ldots \cdot \lambda_k|.$$

This observation was powerful, not just because it gives us a new way to calculate the positive determinant (find elementary matrix decomposition, look for multiply by lambda matrices, take products, win) but because it also gave us a useful tool for proving theorems about the determinant! For example:

- **Theorem.** Take any two $n \times n$ matrices A, B. Then

$$\det{}^+(A \cdot B) = \det{}^+(A) \cdot \det{}^+(B).$$

Examples can be found in: Lecture 19.

- Finally, we ended in week 10 with a brief discussion of the general, or signed **determinant**:

Definition. The **determinant** (as opposed to the "positive determinant") of a matrix A is defined as follows:

(a) Take A, and write it as the product $E_1 \cdot \ldots \cdot E_n$ of elementary matrices.
(b) To find the determinant $\det(A)$ of A, look at these elementary matrices. Let $\lambda_1, \ldots \lambda_k$ denote the constants that show up in the "multiply an entry by λ_i" elementary matrices, and l denote the number of "swap" elementary matrices. Then

$$\det(A) = (-1)^l \cdot \lambda_1 \cdot \ldots \cdot \lambda_k$$

Basically, this is the positive determinant, except we keep track of the signs of the λ's and think of the swaps as contributing factors of -1.

We didn't have a lot of time to do things with this. However, we noted that the theorems we had for the positive determinant have similar results for the actual determinant:

–
– **Theorem.** Take any matrix A. Look at the matrix $A \cdot E$. Then,

$$E_{\text{multiply entry } k \text{ by } \lambda}.$$

Then

* If $E = E_{\text{multiply entry k by } \lambda}$, then $\det(AE) = \lambda \cdot \det(A)$.
* If $E = E_{\text{switch entry } k \text{ and entry } l}$, then $\det(AE) = -\det(A)$.
* If $E = E_{\text{add } \lambda \text{ copies of entry } k \text{ to entry } l}$, then $\det(AE) = \det(A)$.
* **Theorem.** Take any two $n \times n$ matrices A, B. Then

$$\det(A \cdot B) = \det(A) \cdot \det(B).$$

We also had a theorem that related the determinant to the positive determinant:

Theorem. Take any $n \times n$ matrix A. Then, we have

$$|\det(A)| = \det{}^+(A).$$

Finally, one special property we noticed about the determinant was n-linearity. We didn't have much time to work with this, so I don't expect people to know how to use it! However, we did state it in class, and it's possible that a true-false question could involve n-linearity as a concept.

Definition. Let T be a map from $n \times n$ matrices to $\mathbb{R}$. We say that T is **n-linear** if the following always holds:

- Take any matrix A, with columns $\vec{a_{c_1}}, \ldots \vec{a_{c_n}}$.
- Suppose that $\vec{a_{c_i}}$ is equal to some sum of vectors $\vec{x} + \vec{y}$.
- Then, consider the two matrices created by replacing this i-th column with the vectors $\vec{x}, \vec{y}$ respectively:

$$A_x = \begin{bmatrix} \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vec{a_{c_1}} & \ldots & \vec{a_{c_{i-1}}} & \vec{x} & \vec{a_{c_{i+1}}} & \ldots & \vec{a_{c_n}} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \end{bmatrix},$$

$$A_y = \begin{bmatrix} \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vec{a_{c_1}} & \ldots & \vec{a_{c_{i-1}}} & \vec{y} & \vec{a_{c_{i+1}}} & \ldots & \vec{a_{c_n}} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \end{bmatrix},$$

A map is called **n-linear** if

$$T(A) = T(A_x) + T(A_y),$$

for any column $\vec{a_{c_i}}$ and pair of vectors $\vec{x}, \vec{y}$ such that $\vec{x} + \vec{y} = \vec{a_{c_i}}$.

Examples can be found in: Lecture 19.