Math 108a	Professor: Padraic Bartlett

Homework 2: Changes of Basis

Due Wednesday, January 22, by 1:30pm	*UCSB 2014*

Homework problems need to show work and contain proofs in order to receive full credit. Simply stating an answer is only half of the problem in mathematics; you also need to include an argument that persuades your audience that your answer is correct! As always, if you have any questions, feel free to contact either Yihan or I via email or office hours. Have fun!

1. (a) Take the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (x + y, y + z, x + z).$$

Write it as a matrix using the basis

$$B = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}.$$

(b) Take the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$T(w, x, y, z) = (w + x + y, z, w - x - y, z).$$

Write it as a matrix using the basis

$$B = \{(1, 1, 1, 1), (1, 1, 0, 0), (1, -1, 0, 0), (0, 0, 1, -1)\}.$$

2. (a) Consider the identity transformation $id : \mathbb{R}^n \to \mathbb{R}^n$, that sends every vector to itself: i.e.

$$id(\vec{x}) = \vec{x},$$

for any $\vec{x} \in \mathbb{R}^n$. Prove that for **any** basis $B = \{\vec{b_1}, \ldots \vec{b_n}\}$, the matrix associated to id in the base B is always the identity matrix: i.e.

$$id_{\text{written as a matrix in base } B} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}_{B.}$$

(b) In part (b), you showed that the identity transformation is represented as the same matrix under any basis. Are there any other linear transformations $\mathbb{R}^n \to \mathbb{R}^n$ that have this property — i.e. that are written as the same matrix no matter what base you pick? Either find one such matrix and show it has this property, or prove no such matrices exist.

3. Let T be the linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ corresponding to the matrix

$$T_{\text{standard}} = \begin{bmatrix} -7 & 12 & 0 \\ -6 & 10 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Also, let B denote the basis $\{\vec{b_1} = (3,2,0), \vec{b_2} = (4,3,0), \vec{b_3} = (0,0,1)\}$ for $\mathbb{R}^3$.

(a) Write T as a matrix T_B under the basis B.

(b) In class, we took any T with associated matrices T_{standard}, T_B under the standard basis and B-basis, and proved that

$$T_{\text{standard}} = \begin{bmatrix} \vdots & & \vdots \\ \vec{b_1} & \dots & \vec{b_n} \\ \vdots & & \vdots \end{bmatrix} \cdot T_B \cdot \begin{bmatrix} \vdots & & \vdots \\ \vec{b_1} & \dots & \vec{b_n} \\ \vdots & & \vdots \end{bmatrix}^{-1}.$$

Verify this claim for our matrices here: i.e. show that

$$\begin{bmatrix} -7 & 12 & 0 \\ -6 & 10 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots \\ \vec{b_1} & \vec{b_2} & \vec{b_3} \\ \vdots & \vdots & \vdots \end{bmatrix} \cdot T_B \cdot \begin{bmatrix} \vdots & \vdots & \vdots \\ \vec{b_1} & \vec{b_2} & \vec{b_3} \\ \vdots & \vdots & \vdots \end{bmatrix}^{-1}.$$

4. Consider the linear transformation $T(x,y) = (x+y, y)$.

(a) Write this linear transformation in the standard basis.

(b) Show that there are no values of a, b, c, d, e, f such that both of the following conditions hold: (1) the equation

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$$

holds, and (2) the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is also invertible.

(c) Using the above two parts, explain why there is no basis in which T can be written as a diagonal matrix[1].

[1] This might be a little surprising, as we've written several matrices in this form BDB^{-1}, for D a diagonal matrix and B a matrix made out of the elements of some basis. This is because we've been studying examples that are "well-behaved" in some sense; this is an early exposure to the kind of "defective" matrices that are also out there in the world!