Math 108a	Professor: Padraic Bartlett

Homework 3: Orthogonality

Due Monday, January 27, by 1:30pm — *UCSB 2014*

Homework problems need to show work and contain proofs in order to receive full credit. Simply stating an answer is only half of the problem in mathematics; you also need to include an argument that persuades your audience that your answer is correct! If you have any questions, feel free to contact either Yihan or I via email or office hours. Have fun!

1. Find an orthonormal basis for the space

$$V = \{(w, x, y, z) \mid w + x + y + z = 0\}.$$

2. (a) Find the three distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$ for the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 2 \end{bmatrix}$, and their associated eigenvectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$.

   (b) Set $\vec{r_1} = \vec{v_1}/||\vec{v_1}||, \vec{r_2} = \vec{v_2}/||\vec{v_2}||, \vec{r_3} = \vec{v_3}/||\vec{v_3}||$. Prove that the set $R = \{\vec{r_1}, \vec{r_2}, \vec{r_3}\}$ is an orthonormal basis for $\mathbb{R}^3$.

   (c) Write A as a matrix under the basis R. What kind of matrix is A_R?

3. (a) Can you find four nonzero vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ such that

   - $\vec{a}$ is orthogonal to $\vec{b}$,
   - $\vec{b}$ is orthogonal to $\vec{c}$,
   - $\vec{c}$ is orthogonal to $\vec{d}$,
   - $\vec{d}$ is orthogonal to $\vec{a}$,

   but $\vec{a}$ is **not** orthogonal to $\vec{c}$? Either find such a set of four vectors and demonstrate they have these properties, or prove no such set can exist.

   (b) Can you find five nonzero vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e} \in \mathbb{R}^2$, such that

   - $\vec{a}$ is orthogonal to $\vec{b}$,
   - $\vec{b}$ is orthogonal to $\vec{c}$,
   - $\vec{e}$ is orthogonal to $\vec{a}$?
   - $\vec{c}$ is orthogonal to $\vec{d}$,
   - $\vec{d}$ is orthogonal to $\vec{e}$, and

4. In class, our definition of **orthogonality** came from our idea of what a **dot product** was! Surprisingly, however, there are many different notions of dot product. Consider the following definition: for $A =$ the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and any two vectors $\vec{a}, \vec{b} \in \mathbb{R}^2$, define the **inner product of $\vec{x}$ and $\vec{y}$ with respect to** A, $\langle \vec{a}, \vec{b} \rangle_A$, as follows:

$$\langle \vec{a}, \vec{b} \rangle_A = (A \cdot \vec{a}) \cdot \vec{b}.$$

For example, the inner product of $(1,2),(3,4)$ is

$$\langle (1,2),(3,4)\rangle_A = \left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) \cdot (3,4) = (4,5)\cdot(3,4) = 32.$$

We can use this inner product to define a "new" notion of orthogonality: we say that two vectors $\vec{a},\vec{b} \in \mathbb{R}^2$ are **orthogonal with respect to the inner product given by** A if $\langle \vec{a},\vec{b}\rangle_A$ is 0.

(a) Find six distinct nonzero vectors $\vec{a_1},\ldots \vec{a_6}$ in $\mathbb{R}^2$, none of which are scalar multiples of each other, such that the inner products $\langle \vec{a_1},\vec{a_2}\rangle_A, \langle \vec{a_3},\vec{a_4}\rangle_A, \langle \vec{a_5},\vec{a_6}\rangle_A$ are all zero.

(b) Under the normal notion of dot product and orthogonality, two vectors were orthogonal if the angle between them was a right angle. Is that true for the inner product with respect to A? Why?

(c) Suppose I told you that two nonzero vectors $\vec{a},\vec{b}$ met at an angle greater than $120°$. Is it possible that $\langle \vec{a},\vec{b}\rangle_A = 0$? Either prove that no such pair can exist, or create a pair that meet at such an angle and have dot product equal to 0.

5. In Math 108A, we ran into the idea of **polynomials** as vector spaces. Specifically, consider the set $\mathcal{P}_3(\mathbb{R})$, the collection of all polynomials of degree 3 or less: i.e.

$$\mathcal{P}_3(\mathbb{R}) = \{a + bx + cx^2 + dx^3 \mid a,b,c,d \in \mathbb{R}\}.$$

This is a vector space! Consider the following notion of "dot product" for these polynomials: for any pair of polynomials $p(x), q(x) \in \mathcal{P}_3(\mathbb{R})$, we define their **inner product**, $\langle p(x), q(x)\rangle$, as follows:

$$\langle p(x), q(x)\rangle = \int_{-1}^{1} p(x)\cdot q(x)\ dx.$$

For example, the inner product of $1+x$ and $x+x^2$ is the quantity

$$\langle 1+x, x+x^2\rangle = \int_{-1}^{1} (1+x)(x+x^2)\ dx = \int_{-1}^{1} x + 2x^2 + x^3\ dx$$
$$= \left(\frac{x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4} \right)\Bigg|_{-1}^{1} = \frac{4}{3}$$

We can use this inner product to define a notion of **orthogonality**: we say that two polynomials $p(x), q(x)$ are orthogonal if and only if their inner product $\langle p(x), q(x)\rangle$ is 0.

(a) Warmup: choose four nonzero polynomials $p_1(x), q_1(x), p_2(x), q_2(x)$, and calculate the dot products $\langle p_1(x), q_1(x)\rangle, \langle p_2(x), q_2(x)\rangle$.

(b) Find a basis for $\mathcal{P}_3(\mathbb{R})$ that is made of orthogonal polynomials. (Note: because $\mathcal{P}_3(\mathbb{R})$ is a four-dimensional space, as you have four degrees of freedom in the polynomial $a + bx + cx^2 + dx^3$, you should get four polynomials in your basis.)