Math 108B	Professor: Padraic Bartlett

Homework 4: QR and Schur Decompositions

Due Monday, February 10, by 1:30pm — *UCSB 2014*

Homework problems need to show work and contain proofs in order to receive full credit. Simply stating an answer is only half of the problem in mathematics; you also need to include an argument that persuades your audience that your answer is correct! If you have any questions, feel free to contact either Yihan or I via email or office hours. Have fun!

1. (a) Find a QR decomposition of the following matrix:

$$A = \begin{bmatrix} 2 & 2 & -4 & -1 \\ 3 & 9 & 9 & 7 \\ 6 & 3 & 5 & 5 \\ 0 & 2 & 5 & 11 \end{bmatrix}$$

   (b) Use A's QR-decomposition to find a vector $\vec{v}$ such that

$$A\vec{v} = (-1, 28, 19, 18).$$

2. (a) Find the Schur decomposition of the following matrix:

$$A = \begin{bmatrix} 37 & -9 \\ 16 & 13 \end{bmatrix}$$

   (b) Use A's Schur decomposition to find A^{50}.
   (When finding your answer, do not expand large exponents: i.e. write 9^{50}, instead of 515377520732011331036461129765621272702107522001.)

3. Consider the following Markov process, used to model a drunkard's walk:

   - States: there are four states. The first corresponds to the drunkard's home H_1; the second and third correspond to a pair of city blocks C_2, C_3 near the drunkard's walk, and the fourth corresponds to a black hole B_4 that absorbs everything that touches it.
   - Rules:
     - If the drunkard is home, it falls asleep and stays home: i.e. it stays at the first home state H_1 with probability 1, and goes to the other three states with probability 0.
     - If the drunkard is at block C_2, it goes home with probability 1/2, goes to block C_3 with probability 1/2, and never goes to the other states (i.e. goes to them with probability 0.)
     - If the drunkard is at block C_3, it goes to block C_2 with probability 1/2, stumbles into the black hole B_4 with probability 1/2, and never goes to the other states.

- If the drunkard has fallen into the black hole, then it is[1] trapped in the black hole forever: i.e. it stays at the black hole state B_4 with probability 1 and goes to the other 3 states with probability 0.

(a) Create a matrix A that corresponds to this process.

(b) Suppose that our drunk starts at C_3. Over a long enough period of time, what is the likelihood that the drunk makes it home?

4. Suppose that A is a $n \times n$ real-valued symmetric matrix, Q is an orthonormal basis of $\mathbb{R}^n$, and R is an upper-triangular matrix such that A's Schur decomposition is given by the product QRQ^{-1}. Prove that R is a diagonal matrix.

5. Prove or disprove: there is a matrix A such that A is diagonal in some basis, but in A's Schur decomposition QRQ^{-1} the matrix R is not diagonal.

Bonus! If you solve this problem, email your answers to me directly; answers sent to the TA only will not be graded.

Generalize the drunkard's walk to a Markov chain on $\mathbb{Z}^2$, the integer lattice, as follows:

- States: points (i, j), with $i, j \in \mathbb{Z}$.
- Rules: from any point $(i, j) \neq (0, 0)$, your drunkard will move to one of the four points $(i \pm 1, j)$, $(i, j \pm 1)$, each with probability $1/4$. At $(0, 0)$, however, the drunkard is home, and does not leave (i.e. it stays at $(0, 0)$ with probability 1.)

Suppose that your drunk starts at some point $(x, 0)$ on the x-axis. Over a long enough run, what is the probability that it will make it home (i.e. get to $(0, 0)$?)

[1]Stephen Hawking may disagree: see http://arxiv.org/abs/1401.5761. Also if you're a physicist and know things about this, find me because I want to understand this paper?