Math 108B — Professor: Padraic Bartlett

Homework 6: Similar/Elementary Matrices; the Spectral Theorem

Due Wednesday, February 26, by 1:30pm — *UCSB 2014*

Homework problems need to show work and contain proofs in order to receive full credit. Simply stating an answer is only half of the problem in mathematics; you also need to include an argument that persuades your audience that your answer is correct! If you have any questions, feel free to contact either Yihan or I via email or office hours. Have fun!

1. Suppose that A and B are a pair of 2×2 matrices, such that λ is an eigenvalue of A if and only if it is an eigenvalue of B. Is A similar to B? Either create a counterexample to this claim, or prove that it is true.

2. (a) Prove that if A and B are similar, then the trace of A is the trace of B.

   (b) Prove that if A and B are similar, then the determinant of A is the determinant of B.

3. (a) Prove the following claim: the only matrix that is similar to the identity matrix is the identity matrix.

   (b) Is there any other matrix A such that the only matrix similar to A is A itself? Come up with a very simple property characterizing all such matrices. Prove that your property indeed characterizes these kinds of matrices: i.e. prove that any matrix that satisfies your property is only similar to itself, and that if a matrix is only similar to itself then it satisfies your property.

4. Consider the following elementary matrix:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{bmatrix}$$

   Find an upper-triangular matrix that is similar to E.

(Bonus!) If you solve this problem, email your answers to me directly; answers sent to the TA only will not be graded.

Take the field $\mathbb{Z}/2\mathbb{Z} = \langle \{0,1\}, +, \cdot \rangle$, given by the usual concept of modular arithmetic mod 2. With this field, we can define vectors in $(\mathbb{Z}/2\mathbb{Z})^n$, like $(0,1) \in (\mathbb{Z}/2\mathbb{Z})^2$, and matrices like $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ that act on these vectors; as well, we can define the concepts of matrix multiplication mod 2 (i.e. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mod 2$.) For any n, call the collection of $n \times n$ matrices with entries from this field $M_n(\mathbb{Z}/2\mathbb{Z})$.

For any n, what is the largest set of $n \times n$ matrices in $M_n(\mathbb{Z}/2\mathbb{Z})$ that you can find, such that no two of these matrices are similar via some third element in $M_n(\mathbb{Z}/2\mathbb{Z})$?