Math 108B	Professor: Padraic Bartlett

Homework 8: Jordan Canonical Form

Due Friday, March 14, by 1:30pm — *UCSB 2014*

Homework problems need to show work and contain proofs in order to receive full credit. Simply stating an answer is only half of the problem in mathematics; you also need to include an argument that persuades your audience that your answer is correct! If you have any questions, feel free to contact either Yihan or I via email or office hours. Have fun!

1. This problem has you go through an example run of our Jordan canonical form construction. Consider the following matrix:

$$A = \begin{bmatrix} 14 & -6 & -4 & 0 \\ 6 & 2 & 0 & -4 \\ -2 & 2 & 4 & 0 \\ -2 & 2 & 0 & 4 \end{bmatrix}$$

   (a) Find a Schur decomposition $A = URU^{-1}$.

   (b) Now, conjugate R by elementary matrices until it is an upper-triangular matrix R' with the following properties:
   - Whenever $r_{ii} \neq r_{jj}$, $r_{ij} = 0$.

   (c) Conjugate R' by a permutation matrix so that we get an upper-triangular matrix R'' whose diagonal is the same as R, but in order.

   (d) Finally, conjugate R'' by elementary matrices so that it is a block-diagonal matrix B, the blocks of which are all Jordan blocks. Conclude that A can be written in Jordan normal form.

2. (a) Suppose that B is a $n \times n$ Jordan block of the form

$$B = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & \dots & 0 & \lambda \end{bmatrix}$$

   Prove that B is similar to B^T.

   (b) Using 1, prove that if A is a block-diagonal matrix where each block is a Jordan block, then A is similar to A^T.

   (c) Using 2, prove that if A is any matrix, A is similar to A^T.

3. Again, suppose that B is a $n \times n$ Jordan block of the form

$$B = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & \dots & 0 & \lambda \end{bmatrix}$$

(a) Show that λ is the only eigenvalue of B. Let E_λ denote the collection of all vectors $\vec{v}$ such that $B\vec{v} = \lambda\vec{v}$. What is the dimension of E_λ?

(b) Show that there is some value of k such that $(B - \lambda I)^k$ is the all-zeroes matrix. Find the smallest value of k for which this is true.

(c) Suppose that C is any $n \times n$ matrix, such that C has only one distinct eigenvalue λ. Prove that there is some value of k such that $(C - \lambda I)^k$ is the all-zeroes matrix.

4. Suppose that B is a 4×4 Jordan block of the form

$$B = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}.$$

Suppose that $|\lambda| < 1$. Prove that

$$\lim_{k \to \infty} B^k = \text{ the all-zeroes matrix.}$$