Math 137a	Professor: Padraic Bartlett

Lecture 5: Mycielski Graphs

Week 3	*UCSB 2014*

In class on Tuesday, we proved (with little effort) that for any graph G, we have

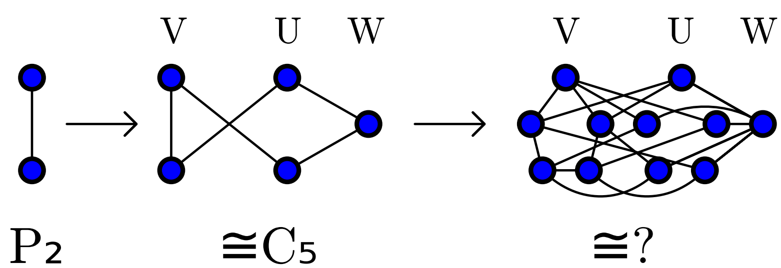
$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1.$$

We noticed that for bipartite graphs, there was often a very big gap between $\Delta(G)+1$ and $\chi(G)$. With the second half of class, we studied something called the **Mycielski construction**, which also showed that there was a big gap between $\omega(G)$ and $\chi(G)$! This isn't in your text, so I wrote up what we did here for reference's sake.

Example. The **Mycielski** construction is a method for turning a triangle-free graph with chromatic number k into a larger triangle-free graph with chromatic number $k+1$. It works as follows:

- As input, take a triangle-free graph G with $\chi(G) = k$ and vertex set $\{v_1, \ldots v_n\}$.
- Form the graph G' as follows: let $V(G') = \{v_1, \ldots v_n\} \cup \{u_1, \ldots u_n\} \cup \{w\}$.
- Start with $E(G') = E(G)$.
- For every u_i, add edges from u_i to all of v_i's neighbors.
- Finally, attach an edge from w to every vertex $\{u_1, \ldots u_n\}$.

Starting from the triangle-free 2-chromatic graph K_2, here are two consecutive applications of the above process:



Proposition 1. *The above process does what it claims: i.e. given a triangle-free graph with chromatic number k, it returns a larger triangle-free graph with chromatic number $k+1$.*

Proof. Let G, G' be as described above. For convenience, let's refer to $\{v_1, \ldots v_n\}$ as V and $\{u_1, \ldots u_n\}$ as U. First, notice that there are no edges between any of the elements in U in G'; therefore, any triangle could not involve two elements from U. Because G was triangle-free, it also could not consist of three elements from V; finally, because w is not connected to any elements in V, no triangle can involve w. So, if a triangle exists, it must consist of two elements v_i, v_j in V and an element u_l in U; however, we know that u_l's only neighbors in V are the neighbors of v_l. Therefore, if (v_i, v_j, u_l) was a triangle, (v_i, v_j, v_l) would also be a triangle; but this would mean that G contained a triangle, which contradicts our choice of G.

Therefore, G' is triangle-free; it suffices to show that G' has chromatic number $k+1$.

To create a proper $k+1$-coloring of G': take a proper coloring $f : V(G) \to \{1, \ldots k\}$ and create a new coloring map $f' : V(G') \to \{1, \ldots k+1\}$ by setting

- $f'(v_i) = f(v_i)$,
- $f'(u_i) = f(v_i)$, and
- $f(w) = k+1$.

Because each u_i is connected to all of v_i's neighbors, none of which are colored $f(v_i)$, we know that no conflicts come up there; as well, because $f(w) = k+1$, no conflicts can arise there. So this is a proper coloring.

Now, take any k-coloring g of G': we seek to show that this coloring must be improper, which would prove that G' is $k+1$-chromatic. To do this: first, assume without any loss of generality that $f(w) = k$ (it has to be colored something, so it might as well be k.)

Then, because w is connected to all of U, the elements of U must be colored with the elements $\{1, \ldots k-1\}$. Let $A = \{v_i \in V : g(v_i) = k\}$. We will now use U to recolor these vertices with the colors $\{1, \ldots k-1\}$: if we can do this properly, then we will have created a $k-1$ proper coloring of G, a k-chromatic graph (and thus arrived at a contradiction.)

To do this recoloring: change g on the elements of A so that $g(v_i)$'s new color is $g(u_i)$. We now claim that g is a proper $k-1$ coloring of G itself. To see this: take any edge $\{v_i, v_j\}$ in G. If both of $v_i, v_j \notin A$, then we didn't change the coloring of v_i and v_j; so this edge is still not monochromatic, because g was a proper coloring of G'. If $v_i \in A$ and $v_j \notin A$, then v_j is a neighbor of v_i and thus (by construction) u_i has an edge to v_j. But this means that $g(u_i) \neq g(v_j)$, because g was a proper coloring of G': therefore, this edge is also not monochromatic!

Because there are no edges between elements of A (as they were all originally colored k under g, and therefore there weren't any edges between them,) this covers all of the cases: so we've turned g into a $k-1$ coloring of a k-chromatic graph. As this is impossible, we can conclude that g cannot exist – i.e. G' cannot be k-colored! So $\chi(G') = k+1$, as claimed. $\square$

As the example above illustrates, our bounds can (unfortunately) be rather loose: the Mycielskians (graphs acquired by taking P_2 and repeatedly applying the above process) have $\omega(M) = 2$, and yet have arbitrarily high chromatic number.