Math 137B	Professor: Padraic Bartlett

Homework 1: Electrical Networks and Random Walks

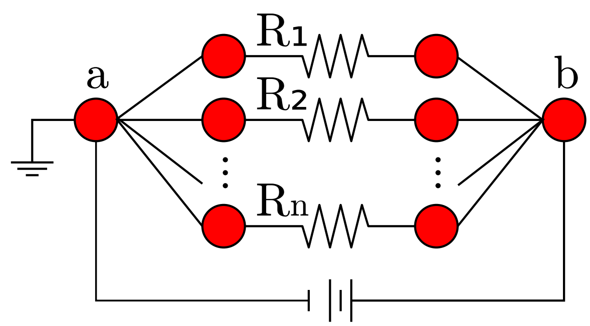
Due Thursday, week 2 — *UCSB 2014*

Pick **three** of the problems in this set to solve! Some are harder than others (i.e. Rayleigh is trickier than most, the resistor laws are pretty simple, etc.), but all problems are worth the same number of points.

1 Theorems From Class

1. Prove the following claims about resistors we made in class:

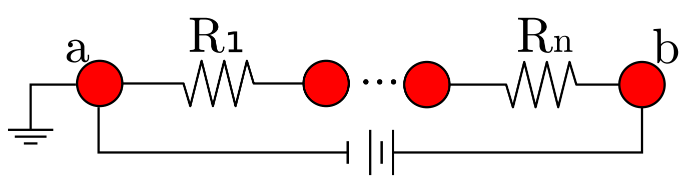
   (a) The effective resistance of the circuit below is the reciprocal of the sum of the reciprocals of the resistors in the circuit. In other words, the circuit

   has effective resistance given by the formula

   $$\frac{1}{R_{\text{eff}}} = \sum_{i=1}^{n} \frac{1}{R_i}.$$

   (b) The effective resistance of the circuit below is the sum of the resistors in the circuit. In other words,

   has effective resistance given by the formula

   $$R_{\text{eff}} = \sum_{i=1}^{n} R_i.$$

2. Prove Rayleigh's Monotonicity Theorem:

Theorem 1. *If any of the individual resistances in a circuit increase, then the overall effective resistance of the circuit can only increase or stay constant; conversely, if any of the individual resistances in a circuit decrease, the overall effective resistance of the circuit can only decrease or stay constant.*

In specific, cutting wires (setting certain resistances to infinity) only increases the effective resistance, while fusing vertices together (setting certain resistances to 0) only decreases the effective resistance.

3. Prove the following lemma we claimed in class:

   Lemma 2. *Suppose C is a circuit with two vertices x, y that are not connected by a resistor and are at the same potential: i.e. $v(x) = v(y)$. Then shorting together x and y does not change the voltages or currents in the circuit.*

2 Calculating Resistances of Things

1. Suppose that we take the 2^n vertices of the n-dimensional cube, connect them all with resistors, ground the origin, and put a $1v$ potential difference between the origin and the point $(1, 1, \ldots 1)$. What is the resistance of this circuit?

2. In class, we proved that p_{esc} on $\mathbb{Z}^3$ was at least $1/6$. The actual value of p_{esc} is actually $\sim .63$. By either finding a different tree, or somehow being clever in another way, improve the bound we came up with in class: show that $p_{\text{esc}} \geq 1/3$.

3. Show that p_{esc} on $\mathbb{Z}^4$ is at least $1/2$.

4. Consider a graph G corresponding to a tiling of $\mathbb{R}^2$ with equilateral triangles with side length 1. Pick some vertex A to denote a starting location, and suppose we model a random walk on this graph starting from A. What is p_{esc}? In particular, is it 0 or nonzero?