Math 137B	Professor: Padraic Bartlett

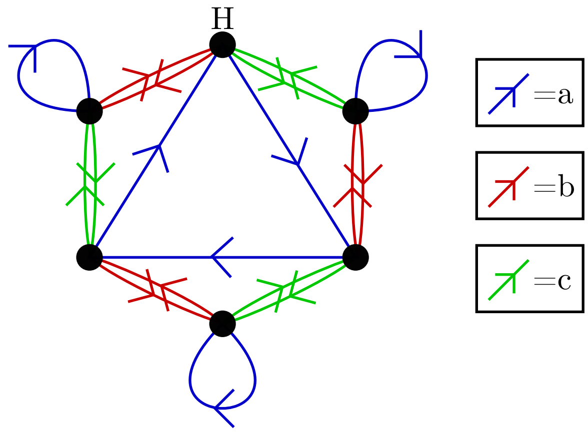
Homework 3: Schreier Graphs

Due Thursday, week 5	*UCSB 2014*

Pick **4** of the problems in this set to solve! Solutions need justification and proof to receive full credit: i.e. it is not enough to simply draw the answer.

1. Let $G = \langle a, b \mid a^3 = id, (ba)^2 = id \rangle$, $H = \langle b \rangle$, and $S = \{a, b\}$. What is the Schreier graph of this group/subgroup/generating set?

2. Suppose that G is a graph generated by three elements a, b, c, and that the diagram below is a Schreier graph for G with respect to some subgroup H and these three generators.



   What is a set of generators for H?

3. Let G be the free group on two generators $\langle a, b \rangle$, and H be the subgroup generated by all of the words in G containing an even number of a's.

   (a) Create the Schreier graph for G with respect to H and the generating set $\{a, b\}$.

   (b) Use this graph to give a set of generators for H. (Notice that the set of generators you get here is a **free** set of generators: i.e. there is no nontrivial combination of these generating words that results in the identity! This is because of the result we stated on Thursday/ finished proving Tuesday.)

4. Let G be the free group on two generators $\langle a, b \rangle$ and let H denote the subgroup consisting of all words of even length. (i.e. $aba^{-1}b^{-1}$ is in H, while aba is not.)

   (a) Show that H is generated by the six words $w_1 = aa, w_2 = bb, w_3 = ab, w_4 = ba, w_5 = ab^{-1}, w_6 = ba^{-1}$.

   (b) Show that this collection of six generators is **not free**: i.e. that there is some relation on these words $w_{i_1}^{k_1} w_{i_2}^{k_2} \ldots w_{i_k}^{k_n} = id$, for some $k_j \in \mathbb{Z}, i_j \in \{1, \ldots n\}$.

   (c) Find a free collection of generators for H.

5. We studied the concept of **vertex-transitivity** on our last problem set, and showed that all Cayley graphs are vertex-transitive. Is every Schreier graph vertex-transitive?

6. On our last problem set, you were asked to show that the dodecahedron's graph is not a Cayley graph. Consider the other platonic solids: i.e. the tetrahedron, the cube, the octahedron, and the icosahedron. Show that all of these can be written as the Cayley graphs of appropriate groups.

7. The Frobenius group of order 20 is the following group:

$$\langle s, t \mid s^4 = t^5 = 1, s^{-1}ts = t^2 \rangle$$

   Alternately, you can consider this as the subgroup of S_5 generated by (2354) and (12345). One set of notes I came across online while researching this HW set claimed that the Cayley graph of this group is the dodecahedron. This sharply contradicts many other sources, which claim the dodecahedron is not the Cayley graph of any group. Find the Cayley graph of this group. Is it the dodecahedron?

8. Take the group $G = S_5$ along with the subgroup $H = S_3 \times S_2 = \{(\pi, \mu) \mid \pi$ is a permutation of $\{1, 2, 3\}$, μ is a permutation of $\{4, 5\}\}$.

   - Explain why the Schreier graph of G with respect to H and any generating set S is a graph on 10 vertices.
   - Pick a generating set S such that this Schreier graph is the Petersen graph.