Math 7h	Professor: Padraic Bartlett

Homework 1: Sizes of Infinity

Due 10/8/13, at the start of class — *UCSB 2013*

Instructions: Choose **one** of the problems below, and work on it until either:

1. You solve the problem, or
2. You have spent about 90 minutes working seriously on the problem.

In either case, you will need to **show all of your work** in order for me to know how you derived your answer/that you've actually spent time on the problem. At the start of our next lecture, you will hand in your work at the start of class. Your work will be graded on the following scale:

1. Full credit: You have demonstrated through your work that you have seriously worked on this problem for 90 minutes.
2. Half credit: You did a pretty poor job of actually showing your work.
3. No credit: A blank assignment, or just absolute nonsense.
4. Bonus half-credit: You actually solved the problem. (Only awarded if work is shown.)

You are welcome to work on more than one problem if you want. If you do so, then your highest score will be used to evaluate your problem set.

These questions are tricky! Email me if you have questions, or attend office hours! Also, feel free to consult outside sources like Wikipedia, other mathematicians in your dorms, etc. Just make sure that all work is written up in your own words and that you fully understand everything you have written.

Homework Problems

1. Can there ever be more words than numbers?

   Specifically: let's suppose that we're limiting ourselves to the 26-character Latin alphabet, and that the only kinds of things that can be **words** are finite strings of characters from the Latin alphabet. So things like

   - rabbit
   - barglearglesnarg
   - ssss
   - froyo

   are all possibly words. Call the set of all possible words $\mathbb{W}$. Is the set $\mathbb{W}$ the same cardinality as $\mathbb{N}$?

2. Suppose that A and B are a pair of sets such that there is an injection $f : A \to B$, and a surjection $g : A \to B$. Is there a bijection from A to B? Either show that such a function exists, or find a counterexample (i.e. two sets A, B such that there is no bijection from A to B, but there is an injection and a surjection from A to B.)

3. Define $\mathbb{Q}$, the set of all **rational numbers**, as follows. Elements of $\mathbb{Q}$ are fractions of the form $\frac{p}{q}$ such that:

   - Both p and q are integers.
   - q is nonzero. We don't want to accidentally divide by 0.
   - p and q have no common factors apart from 1; i.e. $GCD(p, q) = 1$. This rule is meant to make it so that we don't accidentally "repeat" certain fractions: essentially, instead of having things like $\frac{6}{4}$ and $\frac{9}{6}$ in our set, we're choosing to only have fractions written so that the numerator and denominator have no common factors.

   Are $\mathbb{N}$ and $\mathbb{Q}$ the same size? Either construct a bijection from $\mathbb{N}$ to $\mathbb{Q}$, or show that such a bijection is impossible.

4. Given a set A, we say that a set B is a **subset** of A if every member of B is also a member of A. For example, if $A = \{w, x, y, z\}$ and $B = \{w, z\}$, then B is a subset of A, because every member of B is also a member of A. Conversely, the set $C = \{x, y, z, \alpha\}$ is not a subset of A, because the element α is a member of C and not a member of A.

   A special set that bears mentioning is the **empty set**, denoted $\emptyset$. This is the set that contains no elements.

   Given a set A, the **power set** of A, denoted $\mathcal{P}(A)$, is the collection of all of the subsets of A. For example, the power set of $\{1, 2, 3\}$ is the set containing the following eight sets:

   - $\emptyset$
   - $\{1\}$
   - $\{2\}$
   - $\{3\}$
   - $\{1, 2\}$
   - $\{1, 3\}$
   - $\{2, 3\}$
   - $\{1, 2, 3\}$

   We write this as

$$\mathcal{P}(A) = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\Big\}.$$

   (It may seem weird for a set to contain other sets, but this is entirely valid! Sets are just collections of objects, and those objects can be all sorts of strange things, including other sets. If this bothers you, come and talk to me at office hours, or send me an email!)

   (a) Show that the empty set is a subset of every set.

   (b) Calculate the power set of the following three sets:

      i. {thyme, sage}

      ii. {cat, dog, pheasant, quail}

      iii. $\{\alpha, \beta, \gamma, \delta, \epsilon\}$.

      How many elements are in each power set?

   (c) Prove that for any set A, the cardinality of the power set of A, $|\mathcal{P}(A)|$ is greater than the cardinality of A, $|A|$.

5. Create an injection from $\mathcal{P}(\mathbb{N})$ to the real numbers $\mathbb{R}$.