Math/CS 103 Professor: Padraic Bartlett

# Homework 14: Isomorphism and Matrices

*Due *Wednesday,* 11/13/13, at the start of class.* *UCSB 2013*

There are a few sections to this set: a **theoretical** section, a **calculational** section, and a **fun** section. Problems in the theory section are worth **one** point apiece. Problems in the calculational section are worth **half** a point apiece. Problems in the fun section are worth **two** points each, and are fun. Do **four** points worth of problems. Have fun!

## 1 Theory-ish problems

1. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, with associated matrix

$$\begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ \vec{t_{c_1}} & \vec{t_{c_2}} & \dots & \vec{t_{c_n}} \\ \vdots & \vdots & \dots & \vdots \end{bmatrix}.$$

   (The vectors $\vec{t_{c_i}}$ denote the columns of the $m \times n$ matrix associated to $T$.)

   Prove that $\text{null}(T) = \{\vec{0}\}$ if and only if the set of vectors $\{\vec{t_{c_1}}, \vec{t_{c_2}}, \dots, \vec{t_{c_n}}\}$ is linearly independent.

2. Let $T$ be a linear map just like above. Prove that $\text{range}(T) = \mathbb{R}^m$ if and only if the set of vectors $\{\vec{t_{c_1}}, \vec{t_{c_2}}, \dots, \vec{t_{c_n}}\}$ spans $\mathbb{R}^m$.

Suppose, for the moment, that you have proven the above two questions (even if you haven't!) Then, recall the following result from HW #10:

**Theorem.** A linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is an isomorphism if and only if

- $\text{null}(T) = \{\vec{0}\}$.
- $\text{range}(T) = \mathbb{R}^m$.

By sticking these three results together, we get the following result for free:

**Theorem.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, with associated matrix

$$\begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ \vec{t_{c_1}} & \vec{t_{c_2}} & \dots & \vec{t_{c_n}} \\ \vdots & \vdots & \dots & \vdots \end{bmatrix}.$$

Then $T$ is an **isomorphism** — in other words, a linear map that is both injective and surjective — if and only if

- $\{\vec{t_{c_1}}, \vec{t_{c_2}}, \dots, \vec{t_{c_n}}\}$ is linearly independent, and
- $\{\vec{t_{c_1}}, \vec{t_{c_2}}, \dots, \vec{t_{c_n}}\}$ spans $\mathbb{R}^m$.

Even if you didn't prove the above theory problems, you should look at the above theorem carefully! You will need it for the next section.

## 2 Calculational: Find the Isomorphism

There are 8 matrices below. For each, use the criteria given to you on the page earlier to decide if it is an isomorphism.

1. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 3 & 2 & 12 \\ 4 & 2 & 1 \\ 1 & 7 & 7 \\ 7 & 8 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 5 & 8 & 13 & 21 \\ 34 & 55 & 89 & 144 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 2 & 4 \\ 1 & 3 & 6 \\ 9 & 0 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

## 3 Calculational: Invert the Isomorphism

Pick some of the matrices above that corresponded to isomorphisms. For any such matrix $M$, try to find a map $A$ such that $MA = AM =$ the identity matrix. (There are as many problems here as there are isomorphisms in the above section!)

## 4 Fun: now with Putnam Problems

1. Suppose that $C, D$ are $n \times n$ matrices such that $CDCD = 0$. Is it true that $DCDC$ is necessarily equal to 0?

2. Let $A$ be the $4 \times 4$ matrix

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{1,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{1,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{1,4} \end{bmatrix}$$

For any positive integer $k$, define

$$A^{[k]} = \begin{bmatrix} a^k_{1,1} & a^k_{1,2} & a^k_{1,3} & a^k_{1,4} \\ a^k_{2,1} & a^k_{2,2} & a^k_{2,3} & a^k_{1,4} \\ a^k_{3,1} & a^k_{3,2} & a^k_{3,3} & a^k_{1,4} \\ a^k_{4,1} & a^k_{4,2} & a^k_{4,3} & a^k_{1,4} \end{bmatrix}$$

Notice that this is not the same thing as $A^k = \overbrace{A \cdot A \cdot \ldots \cdot A}^{k \text{ times}}$.Suppose that all of the $a_{i,j}$ are real numbers, and that $A^k = A^{[k]}$ for $k = 1, 2, 3, 4$. Prove that $A^k = A^{[k]}$ for all $k \in \mathbb{N}$.