Math/CS 103	Professor: Padraic Bartlett

Homework 16: Orthogonality and Projection

Due 11/22/13, at the start of class. *UCSB 2013*

1 Orthogonality: Review

On HW#5 and the accompanying lecture #6, we introduced the idea of **orthogonality**:

Definition. Take two vectors $(x_1, \ldots x_n), (y_1, \ldots y_n) \in \mathbb{R}^n$. We say that these two vectors are **orthogonal** if their dot product is 0. Alternately, we can say that two vectors are orthogonal if the angle θ between them is $\pm\pi/2$; this is a consequence of a theorem we proved in class, where we showed

$$\vec{x} \cdot \vec{y} = ||\vec{x}|| \cdot ||\vec{y}|| \cos(\theta).$$

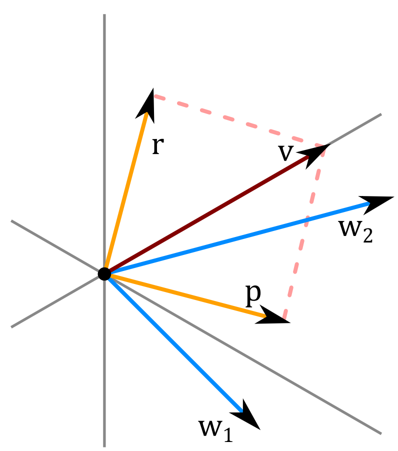
(Recall that $||\vec{x}||$ is the **length** of the vector $\vec{x}$: i.e. the length of $(1, 2, 3)$ is simply the quantity $\sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$.)

2 Orthogonality: The Aim of This Problem Set

This problem set is meant to illustrate a solution to the following problem:

Question 1. *Suppose that we have a collection of vectors $W = \{\vec{w_1}, \ldots \vec{w_k}\}$, and some other vector $\vec{v}$. Is there some way we can write $\vec{v}$ as the sum of two vectors $\vec{r} + \vec{p}$, where $\vec{r}$ is orthogonal to all of the vectors in W, while $\vec{p}$ is contained in the span of W?*

Visually, we're trying to do the following:



The motivation for **why** on earth we want to do this comes later in the problem set; for now, however, have some exercises!

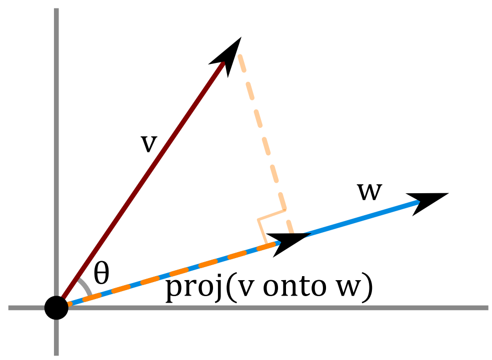
3 Orthogonality: Problems

Do **two problems below.** Even if you do not prove problem 1 or problem 3, you can use it in other problems; in fact, you're going to need them later in this class, so you should definitely give them a look.

1. Take any two vectors $\vec{v}, \vec{w}$. Form the **projection** of $\vec{v}$ onto $\vec{w}$, denoted proj($\vec{v}$ onto $\vec{w}$), as described below:

   - Take the vector $\vec{w}$.
   - Draw a line perpindicular to the vector $\vec{w}$, that goes through the point $\vec{v}$ and intersects the line spanned by the vector $\vec{w}$.
   - proj($\vec{v}$ onto $\vec{w}$) is precisely the point at which this perpindicular line intersects $\vec{w}$.

   We illustrate this below:

   Prove that

   $$\text{proj}(\vec{v} \text{ onto } \vec{w}) = \frac{\vec{v} \cdot \vec{w}}{||\vec{w}||^2} \cdot \vec{w}.$$

2. (a) Calculate the projection of $(1, 1, 1)$ onto $(1, 2, 2)$, using problem 1 above.
   (b) Calculate the projection of $(2, 1, 0)$ onto $(2, 3, 6)$.
   (c) Calculate the projection of $(-1, 1, -1, 1)$ onto $(5, 7, 31, 101)$.

3. Take any two vectors $\vec{v}, \vec{w}$. Consider the vector

   $$\text{orth}(\vec{v} \text{ onto } \vec{w}) = \vec{v} - \text{proj}(\vec{v} \text{ onto } \vec{w}).$$

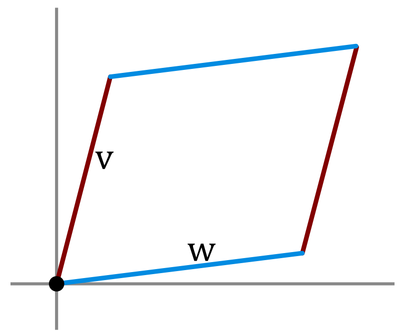
   By taking the dot product, show that orth($\vec{v}$ onto $\vec{w}$) is orthogonal to the vector $\vec{w}$.

4. (a) Calculate orth($\vec{v}$ onto $\vec{w}$), where $\vec{v} = (0, 1, 0)$ and $\vec{w} = (4, 4, 7)$.
   (b) Calculate orth($\vec{v}$ onto $\vec{w}$), where $\vec{v} = (1, 1, 1)$ and $\vec{w} = (3, 4, 12)$.
   (c) Calculate the projection of $(\overbrace{1, 0 \ldots 0}^{25 \text{ entries}})$ onto $(24, 23, 22, \ldots, 2, 1, 0)$.

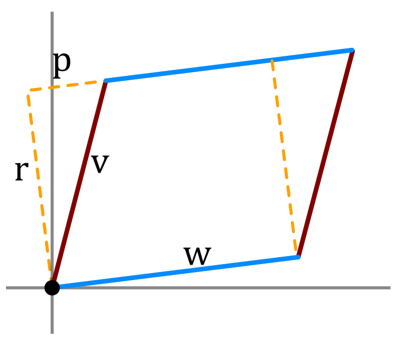
4 Orthogonality: Why Do We Care?

Orthogonality is a concept we talked about a long time ago; why return to it now?

The rough idea for why we care about orthogonality now is because it's the easiest way to understand the idea of **n-dimensional volume**! Specifically: suppose you have a parallelogram spanned by the two vectors $\vec{v}, \vec{w}$.

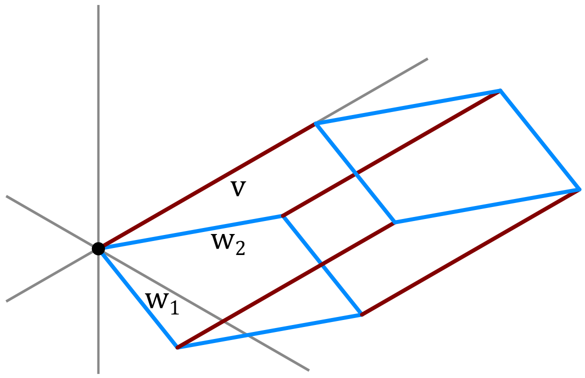


What's the area of this parallelogram? Well, it's the length of the **base** times the **height**, if you remember your high-school geometry! But what are these two quantities? Well: the **base** has length just given by the length of $\vec{w}$. The **height**, however, is precisely the kind of thing we've been calculating in this set! Specifically: suppose that we can write $\vec{v}$ as the sum $\vec{p}+\vec{r}$, where $\vec{p}$ is some multiple of $\vec{v}$ and $\vec{r}$ is orthogonal to $\vec{w}$. Then the length of $\vec{r}$ is precisely the **height**!

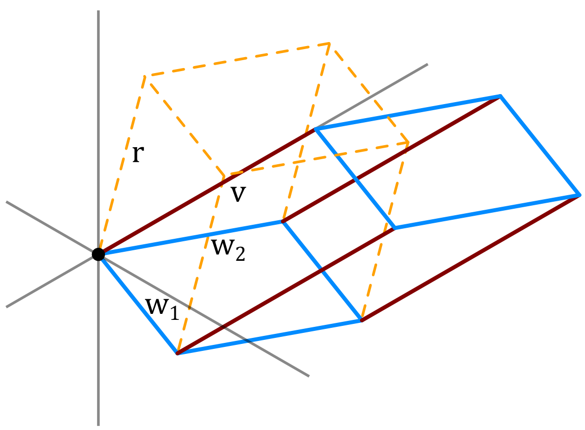


Therefore, to find the area here, we just need to multiply the length of $\vec{r}$ and the length of $\vec{w}$ together.

For three dimensions, the picture is similar. Suppose you want to find the volume of a parallelepiped — i.e. the three-dimensional analogue of a parallelogram – spanned by the three vectors $\vec{v}, \vec{w_1}, \vec{w_2}$.



What's the volume of this parallelotope? Well, this is not much harder to understand than the two-dimensional case: it's just the **area** of the parallelogram spanned by the two vectors $\vec{w_1}, \vec{w_2}$ times the **height**! And again, the height is precisely what we've been studying in this homework. Suppose that we can write $\vec{v} = \vec{r} + \vec{p}$, for some vector $\vec{p}$ in the span of $\vec{w_1}, \vec{w_2}$ and some vector $\vec{r}$ orthogonal to $\vec{w_1}, \vec{w_2}$. Then the length of this vector $\vec{r}$ is, again, precisely the **height**!



This process generalizes to n dimensions: to find the volume of a n-dimensional parallelotope spanned by n vectors $\vec{w_1}, \ldots \vec{w_n}$, we just start with $\vec{w_1}$, and repeatedly for each $\vec{w_2}, \vec{w_n}$, find the "height" of each $\vec{w_i}$ over the set $\vec{w_1}, \ldots \vec{w_{i-1}}$ by doing this "write $\vec{w_i}$ as a projection $\vec{p}$ onto $\{\vec{w_1}, \ldots \vec{w_{i-1}}\}$, plus an orthogonal bit $\vec{r}$, whose length is the height" trick. By taking the product of all of these heights, we get what we would expect to be the n-dimensional volume of the parallelotope! (In fact, it's kinda confusing just what n-dimensional volume even **means**, so if you want you can take this as the **definition** of volume for these kinds of objects in n-dimensional space.)