Math/CS 103 | Professor: Padraic Bartlett

Homework 17: Volume!

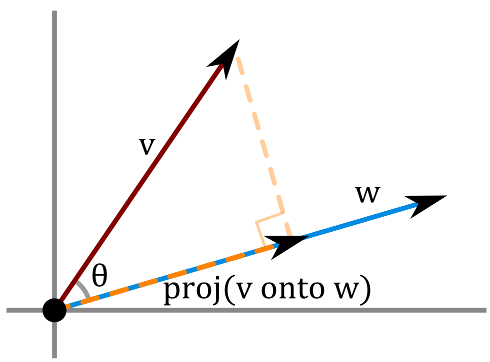
Due 11/25/13, at the start of class. | *UCSB 2013*

On Wednesday we talked about volume! We have some exercises to expand on the ideas in that lecture. Recall the following concepts we defined in the last lecture and previous HW:

Definition. Let $\vec{v}, \vec{w}$ be a pair of vectors in $\mathbb{R}^n$. The **projection** of $\vec{v}$ onto $\vec{w}$, denoted proj($\vec{v}$ onto $\vec{w}$), is the following vector:

- Take the vector $\vec{w}$.
- Draw a line perpindicular to the vector $\vec{w}$, that goes through the point $\vec{v}$ and intersects the line spanned by the vector $\vec{w}$.
- proj($\vec{v}$ onto $\vec{w}$) is precisely the point at which this perpindicular line intersects $\vec{w}$.

We illustrate this below:



On the HW, you proved that

$$\text{proj}(\vec{v} \text{ onto } \vec{w}) = \frac{\vec{v} \cdot \vec{w}}{||\vec{w}||^2} \cdot \vec{w}.$$

Using this concept, we created the vector orth:

$$\text{orth}(\vec{v} \text{ onto } \vec{w}) = \vec{v} - \text{proj}(\vec{v} \text{ onto } \vec{w}).$$

Geometrically, we interpreted the length of this vector as the "height" of $\vec{v}$ of $\vec{w}$. We used this idea of "height" when studying parallelograms:

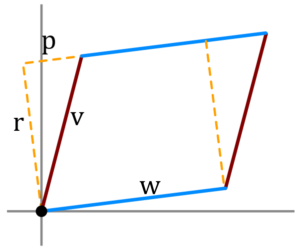
Definition. Take n vectors $\vec{v_1}, \ldots \vec{v_n}$ in $\mathbb{R}^n$. We define the **parallelotope** spanned by these n vectors as the collection of points

$$\text{partope}(\vec{v_1}, \ldots \vec{v_n}) = \{a_1\vec{v_1} + \ldots a_n\vec{v_n} \mid \}$$

If $n = 2$, we call these objects **parallelograms.** If $n = 3$, we call these things **parallelepipeds**.

Using this, we noticed the following property:

Theorem. Let $\vec{v}, \vec{w}$ be two vectors that span a parallelogram. Then the area of this parallelogram is simply the length of the base (i.e. $||\vec{w}||$) times its height (i.e. $||\text{orth}(\vec{v} \text{ over } \vec{w})||$). We draw this below, where $\vec{r}$ is short for $\text{orth}(\vec{v} \text{ over } \vec{w})$.



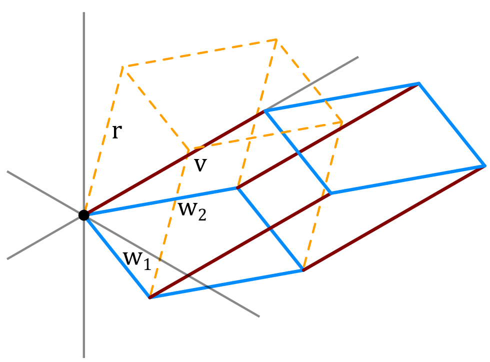
We generalized this to n-dimensions via a tricky construction written up in the notes. For this HW, however, you just need the three-dimensional version:

Theorem. Let $\vec{v}, \vec{w_1}, \vec{w_2}$ be three vectors that span a parallelepiped. Then the volume of this parallelepiped is simply

- the length of one side of the base (i.e. $||\vec{w_1}||$),
- times the height of the other side of the base (i.e. $||\text{orth}(\vec{w_2} \text{ over } \vec{w_1})||$),
- times the height of $\vec{v}$ over the base spanned by the two vectors $\vec{w_1}, \text{orth}(\vec{w_2} \text{ over } \vec{w_1})$. We noted that this was in particular the length of

$$\text{orth}(\vec{v} \text{ over } \vec{w_2}, \vec{w_1}) = \vec{v} - \text{proj}(\vec{v} \text{ onto } \vec{w_1}) - \text{proj}(\vec{v} \text{ onto orth}(\vec{w_2} \text{ over } \vec{w_1})).$$

We draw this below, where $\vec{r}$ is short for $\text{orth}(\vec{v} \text{ over } \vec{w_2}, \vec{w_1})$.



We used all of this work to define the concept of the positive **determinant**:

Definition. Take a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$. The **positive determinant** of this map, i.e. $\det^+(T)$, is the volume of $T(\text{unit cube})$. (The unit cube was just the set $\{(x_1, \ldots x_n) \mid 0 \leq x_i \leq 1, \}$.)

Notice that because T sends each $\vec{e_i}$ to $T(\vec{e_i})$, the $\det^+$ is just the volume of the parallelotope spanned by $T(\vec{e_1}), \ldots T(\vec{e_n})$.

These definitions were pretty crazy. We worked some examples in class, and have other in the notes. This HW is all about getting practice with calculating these things!

1 Problems

There are eight linear maps below, given by their associated matrices. Calculate the positive determinants of **four** of them!

1.
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

2.
$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$$

3.
$$\begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

4.
$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5.
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

6.
$$\begin{bmatrix} 1 & 5 & 5 \\ 5 & 1 & 5 \\ 5 & 5 & 1 \end{bmatrix}$$

7.
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

8.
$$\begin{bmatrix} 0 & k & k \\ k & 0 & k \\ k & k & 0 \end{bmatrix}$$

Alternately: feel free to substitute a **proof** below for any determinant calculation!

1. Take an $n \times n$ matrix A of the form

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Define the **transpose** of this matrix, A^T, as the matrix where we "flip" A over its top left-bottom right diagonal, i.e. where we switch the rows and columns of A, i.e. where we put the entry a_{ji} in the (i,j)-th entry of A^T, i.e.

$$\begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Show that $\det^+(A^T) = \det^+(A)$.

2. Take an arbitrary parallelotope P spanned by the vectors $\{\vec{w_1}, \ldots \vec{w_n}\}$. Suppose that you multiply this parallelotope by an elementary matrix E. What happens to its volume? (There are three kinds of elementary matrices E to consider here.)

3. Show that for any two linear maps with associated matrices A, B, we have

$$\det{}^+(AB) = \det{}^+(A) \det{}^+(B).$$

4. (Putnam, 2002-A-4.) In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?