Math/CS 103	Professor: Padraic Bartlett
Homework 5: The Signal and the Noise	
Due 10/11/13, at the start of class	*UCSB 2013*

Choose **three** of the **six** problems below to complete by next class. Be ready and able to present your solutions if you have them, or your questions if you don't solve the problems!

There is also a bonus question! Complete it[1] to get a free pass on any future HW.

1. Electrical engineering! Specifically, we have the following problem:

   A **signal** of length n, for the purposes of this problem set, is some finite sequence $(s_1, \ldots s_n)$ of $+1$ and -1's. For example, $(+1, -1, -1, -1)$ is a signal of length 4.

   Given a signal A of length n, for any j between 0 and $n-1$, we define the **autocorrelation coefficients**[2] of this sequence to be the values

   $$c(A, j) = |a_1 a_{1+j} + a_2 a_{2+j} + \ldots + a_{n-j} a_n| \,.$$

   (a) Prove that for any signal A, $c(A, 0)$ is just the length of the signal.

   (b) Show that for any j between 1 and 3, $(+1, +1, -1, +1)$ has $c(A, j) \leq 1$.

   (c) Find a signal A of length 5 such that for any n between 1 and 4, we have $c(A, j) \leq 1$.

2. Find a signal A of length 11 such that for any n between 1 and 10, we have $c(A, j) \leq 1$. (Fun fact: this is how your wireless actually works under the 802.11b standard.)

3. We can generalize this notion of correlation from problem 1 to vectors! In particular, take any two strings $\vec{a} = (a_1, \ldots a_n)$, $\vec{b} = (b_1, \ldots b_n)$ of real numbers. Define the correlation of these two vectors to be the quantity

   $$c(\vec{a}, \vec{b}) = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

   (a) Find three vectors $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^2$ such that $c(\vec{u}, \vec{v}), c(\vec{v}, \vec{w}), c(\vec{w}, \vec{u})$ are all negative.

   (b) Find four vectors $\vec{u}, \vec{v}, \vec{w}, \vec{x}$ in $\mathbb{R}^3$ such that $c(\vec{u}, \vec{v}), c(\vec{v}, \vec{w}), c(\vec{w}, \vec{x}), c(\vec{x}, \vec{u})$ are all negative.

[1] Warning: this question is maybe just a little tiny bit tricky. Just a little.

[2] The idea here is basically the following: suppose you have two devices trying to communicate wirelessly on some radio wavelength. You could have each device simply send signals of 1's and 0's to the other on that wavelength: but what if there's interference? Like, if there were occasionally bits of static that would get in the way, you'd lose vital bits of information. Similarly, if some other devices started transmitting on your wavelength, you might not be able to tell this "cross-traffic" apart from the device you're actually trying to communicate with.

A solution is to use these codes! Specifically: let A be a solution to #1(b). Every time you want to send a 1, instead send one copy of our signal A. Every time you want to send a 0, send -1 times a copy of A. On the receiving end, we simply take all incoming signals $(s_1, \ldots s_n)$, and correlate these signals with A: i.e. look at $|s_1 a_1 + \ldots s_n a_n|$. If this number is large and close to $\pm n$, then it's likely a copy of our original signal, and we should interpret this as a ± 1 sent from the device we're communicating with! If it's small, it's likely some cross-traffic or just static in the air, and we can ignore it.

4. Prove that there is no set of five vectors in $\mathbb{R}^3$, such that the correlation of any two of these vectors is negative.

5. Find three vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ with the following properties:

   - $c(\vec{u}, \vec{v}), c(\vec{v}, \vec{w}), c(\vec{w}, \vec{u})$ are all 0.
   - None of the x, y or z-components of any of these vectors are equal to 0. (I.e. $(2, 2, 0)$ is not a valid choice of vector.)

6. Show that you cannot find four nonzero vectors in $\mathbb{R}^3$, such that the correlation of any two of these vectors is 0.

Bonus! 7. Find a signal A of length greater than 15 such that for any n between 1 and 14, we have $c(A, j) \leq 1$.