| Math/CS 103 | Professor: Padraic Bartlett |
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# Lecture 11: Matrix Composition

*Week 6* *UCSB 2013*

In class, we presented several example problems to give us practice with matrix composition! We go over those here.

## 1 Matrix Composition: The Definitions

On HW#13, one of the questions asks you to show the following statement:

**Theorem.** Take any pair of linear maps $A : \mathbb{R}^n \to \mathbb{R}^m, B : \mathbb{R}^m \to \mathbb{R}^k$ with associated matrices

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}, B = \begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,m} \\ b_{2,1} & b_{2,2} & \dots & b_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k,1} & b_{k,2} & \dots & b_{k,m} \end{bmatrix}.$$

Let $\vec{b_{r_i}}$ denote the vector given by the $i$-th row of $B$, and $\vec{a_{c_j}}$ denote the vector given by the $j$-th column of $A$.

Then. the matrix corresponding to their composition $B \circ A : \mathbb{R}^n \to \mathbb{R}^k$ is the $k \times n$ matrix

$$\begin{bmatrix} \vec{b_{r_1}} \cdot \vec{a_1} & \vec{b_{r_1}} \cdot \vec{a_2} & \dots & \vec{b_{r_1}} \cdot \vec{a_n} \\ \vec{b_{r_2}} \cdot \vec{a_1} & \vec{b_{r_2}} \cdot \vec{a_2} & \dots & \vec{b_{r_2}} \cdot \vec{a_n} \\ \dots & \dots & \ddots & \dots \\ \vec{b_{r_k}} \cdot \vec{a_1} & \vec{b_{r_k}} \cdot \vec{a_2} & \dots & \vec{b_{r_k}} \cdot \vec{a_n} \end{bmatrix}.$$

Even if you didn't prove this problem, I've asked people to become comfortable with at least **using** this result, so that they can manipulate and do things with matrix multiplication!

In lecture, I asked a number of problems that were designed to give you some experience with this concept. Proofs of these problems are presented here!

## 2 Examples

**Question.** For any $n$, can you find an $n \times n$ matrix $M$ such that $M^n$ is equal the the all-zeroes matrix, but $M^k$ is not, for any $1 \leq k \leq n-1$?

**Answer.** Yes! Consider the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

This is the matrix with 0's everywhere except for the cells directly above the top left-bottom right diagona, which are 1's. The linear map corresponding to this matrix is the map:

$$T(x_1, \dots x_n) = (0, x_1, \dots x_{n-1}).$$

Notice that composing this map with itself $k$ times gives us the map

$$T^k(x_1, \dots x_n) = \underbrace{T \circ \dots \circ T}_{k \text{ compositions}} = (\underbrace{0, 0, \dots 0}_{k \text{ zeroes}}, x_1, \dots x_{n-k}).$$

In particular, if $k = n$, this map sends every vector to the all-zeroes vector $(0, \dots 0)$. Therefore, the matrix corresponding to this vector is the all-zeroes matrix.

As well, if $k < n$, then this map does not send all vectors to $(0, \dots 0)$. In particular, it sends $e_i$ to $e_{i+k}$, for any $i \leq n - k$, and therefore has corresponding matrix

$$\begin{array}{c} \overbrace{\phantom{0 \quad \dots \quad 0}}^{\text{k zeroes}} \\ \begin{bmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \end{array} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} k \text{ zeroes}$$

which is in particular not the all-zeroes matrix.

**Question.** Can you find a pair of matrices $A, B$ such that $A \cdot B \neq B \cdot A$?

**Answer.** Sure! This property is true for almost any pair of matrices you'd randomly try (as long as you don't like "randomly" choosing the identity matrix.) For one example, just notice that for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

we have

$$A \cdot B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} (1,1,0)\cdot(1,0,1) & (1,1,0)\cdot(0,0,0) & (1,1,0)\cdot(0,1,0) \\ (0,0,1)\cdot(1,0,1) & (0,0,1)\cdot(0,0,0) & (0,0,1)\cdot(0,1,0) \\ (1,0,1)\cdot(1,0,1) & (1,0,1)\cdot(0,0,0) & (1,0,1)\cdot(0,1,0) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \text{ and}$$
$$B \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} (1,0,0)\cdot(1,0,1) & (1,0,0)\cdot(1,0,0) & (1,0,0)\cdot(0,1,1) \\ (0,0,1)\cdot(1,0,1) & (0,0,1)\cdot(1,0,0) & (0,0,1)\cdot(0,1,1) \\ (1,0,0)\cdot(1,0,1) & (1,0,0)\cdot(1,0,0) & (1,0,0)\cdot(0,1,1) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

**Question.** Can you find a pair of matrices $A, B$ such that $A \cdot B \neq 0$, but $B \cdot A = 0$?

**Answer.** Sure! Try

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$A \cdot B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (0,1)\cdot(0,0) & (0,1)\cdot(0,1) \\ (0,0)\cdot(0,0) & (0,0)\cdot(0,1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ and}$$
$$B \cdot A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (0,0)\cdot(0,0) & (0,0)\cdot(1,0) \\ (0,1)\cdot(0,0) & (0,1)\cdot(1,0) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

**Question.** Can you find a pair of matrices $A, B$ such that $B \cdot A$ is the identity matrix, while $A \cdot B$ is not the identity matrix?

**Answer.** Sure! Try

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then

$$A \cdot B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} (1,0)\cdot(1,0) & (1,0)\cdot(0,1) & (1,0)\cdot(0,0) \\ (0,1)\cdot(1,0) & (0,1)\cdot(0,1) & (0,1)\cdot(0,0) \\ (0,0)\cdot(1,0) & (0,0)\cdot(0,1) & (0,0)\cdot(0,0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and}$$
$$B \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (1,0,0)\cdot(1,0,0) & (1,0,0)\cdot(0,1,0) \\ (0,1,0)\cdot(1,0,0) & (0,1,0)\cdot(0,1,0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This trick involves heavily using the fact that these are matrices that map to and from different spaces: $A$ is a matrix that sends vectors in $\mathbb{R}^2$ to $\mathbb{R}^3$, while $B$ is a matrix that sends vectors in $\mathbb{R}^3$ to $\mathbb{R}^2$. If you restrict this to $n \times n$ matrices, however, there's no way to do this trick! We'll discuss why this is in a future class.

**Question.** Can you find an $n \times n$ matrix with integer entries such that the following properties are satisfied?

- The dot product of any row with itself is even.
- The dot product of any two distinct rows is odd.

**Answer.** For odd $n$, this is possible: use the matrix

$$\begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}$$

The dot product of any row with itself is simply the number of 1's in that row, which is $n-1$, which is even. The dot product of any two different rows gives you $n-2$ 1's, because these rows agree in $n-2$ places and have 0's in the remaining two. This is odd; therefore, we satisfy our properties!

For even $n$, things get weirder. It turns out that this is impossible, but it's relatively tricky to show this! You need techniques related to finite fields and vector spaces, which we'll get into next quarter...