Math/CS 103	Professor: Padraic Bartlett

Lecture 12: Elementary Matrices

Week 7 *UCSB 2013*

On HW#13, we studied three special kinds of matrices:

Definition. The first matrix, $E_{\text{multiply entry k by }\lambda}$, is the matrix corresponding to the linear map that multiplies its k-th coördinate by λ and does not change any of the others. On the homework, you showed that it has the form

$$E_{\text{multiply entry } k \text{ by } \lambda} = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \lambda & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

This matrix has 1's down its diagonal and 0's elsewhere, with an exception for the value at (k,k), which is λ.

The second matrix, $E_{\text{switch entry } k \text{ and entry } l}$, corresponds to the linear map that swaps its k-th coördinate with its l-th coördinate, and does not change any of the others. On the homework, you showed that it has the form

$$E_{\text{switch entry } k \text{ and entry } l} = \begin{bmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \ldots & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 1 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 1 \end{bmatrix}$$

You can create this matrix by starting with a matrix with 1's down its diagonal and 0's elsewhere, and switching the k-th and l-th columns.

Finally, the third matrix, $E_{\text{add } \lambda \text{ copies of entry } k \text{ to entry } l}$, for $k \neq l$, corresponds to the linear map that adds λ copies of its k-th coördinate to its l-th coördinate and does not change any of the others. On the homework, you showed that it has the form

$$E_{\text{add } \lambda \text{ copies of entry } k \text{ to entry } l} = \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & \lambda & 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix}$$

This matrix has 1's down its diagonal and 0's elsewhere, with an exception for the value in row l, column k, which is λ.

These three matrices are called the **elementary** matrices. They're incredibly cool, and we're going to study them in this lecture.

1 Elementary Matrices: What They Do

The first thing we want to talk about is what these matrices **do**! Specifically, take any $n \times n$ matrix A. What is the matrix corresponding to $E_{\text{multiply entry k by } \lambda} \circ A$? What do the other elementary matrices do to A?

We study this in the following theorem:

Theorem 1. *Take any $n \times n$ matrix A. Suppose that we are looking at the composition $E \circ A$, where E is one of our elementary matrices. Then, we have the following three possible situations:*

- *if $E = E_{\textit{multiply entry k by } \lambda}$, then $E \circ A$ would be the matrix A with its k-th row multipled by λ.*
- *if $E = E_{\textit{switch entry k and entry l}}$, then $E \circ A$ would be the matrix A with its k-th and l-th rows swapped, and*
- *if $E = E_{\textit{add } \lambda \textit{ copies of entry k to entry l}}$, then $E \circ A$ would be the matrix A with λ copies of its k-th row added to its l-th row.*

Proof. To prove these claims, we repeatedly use the following result from the homework, that told us how to "compose" or "multiply" two matrices together:

Theorem. HW#13, Problem 2.2 Take any pair of linear maps $A : \mathbb{R}^n \to \mathbb{R}^m, B : \mathbb{R}^m \to \mathbb{R}^k$ with associated matrices

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}, B = \begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,m} \\ b_{2,1} & b_{2,2} & \dots & b_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k,1} & b_{k,2} & \dots & b_{k,m} \end{bmatrix}.$$

Look at the linear map given by the composition of these two maps: i.e. consider the linear map $B \circ A : \mathbb{R}^n \to \mathbb{R}^k$. Denote the row vectors of B as $\vec{b_{r_i}}$'s and the column vectors of A as $\vec{a_{c_j}}$'s. We claim that this linear map corresponds to the $k \times n$ matrix

$$\begin{bmatrix} \vec{b_{r_1}} \cdot \vec{a_{c_1}} & \vec{b_{r_1}} \cdot \vec{a_{c_2}} & \ldots & \vec{b_{r_1}} \cdot \vec{a_{c_n}} \\ \vec{b_{r_2}} \cdot \vec{a_{c_1}} & \vec{b_{r_2}} \cdot \vec{a_{c_2}} & \ldots & \vec{b_{r_2}} \cdot \vec{a_{c_n}} \\ \ldots & \ldots & \ddots & \ldots \\ \vec{b_{r_k}} \cdot \vec{a_{c_1}} & \vec{b_{r_k}} \cdot \vec{a_{c_2}} & \ldots & \vec{b_{r_k}} \cdot \vec{a_{c_n}} \end{bmatrix}.$$

In other words, to get the matrix given by composing two matrices, we simply dot the rows of the first matrix with the columns of the second matrix in the manner described above.

Given this result, we simply calculate $E \circ A$ for each of the three cases we've described above.

To start, take any $n \times n$ matrix A, row k and constant λ, and examine the product

$$E_{\text{multiply entry } k \text{ by } \lambda} \circ A$$
$$= \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \lambda & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{bmatrix} \circ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & \ldots & a_{3n} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & \ldots & a_{4n} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & \ldots & a_{5n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & a_{n5} & \ldots & a_{nn} \end{bmatrix}.$$

What do entries in the resulting matrix look like? Well, there are two cases:

- in the location (i, j), for any $i \neq k$ and any j, we know that the entry there is just the dot product of E's i-th row and A's j-th column: i.e.

  $$\text{entry } (i, j) = (0, \ldots, 1, \ldots 0) \cdot (a_{1j}, \ldots a_{nj}) = a_{ij},$$

  because the 1 in the i-th row of E is in the i-th place.

- in the location (k, j), for any j, we know that the entry there is just the dot product of E's k-th row and A's j-th column: i.e.

  $$\text{entry } (k, j) = (0, \ldots, \lambda, \ldots 0) \cdot (a_{1j}, \ldots a_{nj}) = \lambda_{kj},$$

  because the λ in the k-th row of E is in the k-th place.

By inspection, this matrix is precisely

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & a_{k-1,4} & a_{k-1,5} & \dots & a_{k-1,n} \\ \lambda a_{k1} & \lambda a_{k2} & \lambda a_{k3} & \lambda a_{k4} & \lambda a_{k5} & \dots & \lambda a_{kn} \\ a_{k+1,1} & a_{k+1,2} & a_{k+1,3} & a_{k+1,4} & a_{k+1,5} & \dots & a_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & a_{n5} & \dots & a_{nn} \end{bmatrix}.$$

So this elementary matrix works as claimed.

The proofs for the other two elementary matrices are similar. For the matrix $E_{\text{switch entry } k \text{ and entry } l}$, we again examine the product $E \circ A$:

$$E_{\text{switch entry } k \text{ and entry } l} \circ A$$
$$= \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \circ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & \dots & a_{3n} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & \dots & a_{4n} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & \dots & a_{5n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & a_{n5} & \dots & a_{nn} \end{bmatrix}.$$

Again, what do entries in the resulting matrix look like? In this situation, there are three cases:

- In the location (i, j), for any $i \neq k, l$ and any j, we know that the entry there is just the dot product of E's i-th row and A's j-th column: i.e.

  $$\text{entry } (i, j) = (0, \dots, 1, \dots 0) \cdot (a_{1j}, \dots a_{nj}) = a_{ij},$$

  because the 1 in the i-th row of E is in the i-th place.

- In the location (k, j), for any j, we know that the entry there is just the dot product of E's k-th row and A's j-th column: i.e.

  $$\text{entry } (k, j) = (0, \dots, 1, \dots 0) \cdot (a_{1j}, \dots a_{nj}) = a_{lj},$$

  because the 1 in the k-th row of E is in the l-th place.

- In the location (l, j), for any j, we know that the entry there is just the dot product of E's l-th row and A's j-th column: i.e.

  $$\text{entry } (l, j) = (0, \dots, 1, \dots 0) \cdot (a_{1j}, \dots a_{nj}) = a_{kj},$$

  because the 1 in the l-th row of E is in the k-th place.

By inspection, this matrix is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & a_{k-1,4} & a_{k-1,5} & \dots & a_{k-1,n} \\ a_{l1} & a_{l2} & a_{l3} & a_{l4} & a_{l5} & \dots & a_{ln} \\ a_{k+1,1} & a_{k+1,2} & a_{k+1,3} & a_{k+1,4} & a_{k+1,5} & \dots & a_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{l-1,1} & a_{l-1,2} & a_{l-1,3} & a_{l-1,4} & a_{l-1,5} & \dots & a_{l-1,n} \\ a_{k1} & a_{k2} & a_{k3} & a_{k4} & a_{k5} & \dots & a_{kn} \\ a_{l+1,1} & a_{l+1,2} & a_{l+1,3} & a_{l+1,4} & a_{l+1,5} & \dots & a_{l+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & a_{n5} & \dots & a_{nn} \end{bmatrix}.$$

This is A with its k-th and l-th rows swapped, as claimed.

Finally, we turn to $E_{\text{add } \lambda \text{ copies of entry } k \text{ to entry } k}$, and again look at $E \circ A$:

$$\begin{aligned} & E_{\text{add } \lambda \text{ copies of entry } k \text{ to entry } l} \circ A \\ = & \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & \lambda & 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix} \circ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & \dots & a_{3n} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & \dots & a_{4n} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & \dots & a_{5n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & a_{n5} & \dots & a_{nn} \end{bmatrix}. \end{aligned}$$

Again, what do entries in the resulting matrix look like? In this situation, there are just two last cases:

- In the location (i, j), for any $i \neq l$ and any j, we know that the entry there is just the dot product of E's i-th row and A's j-th column: i.e.

  $$\text{entry } (i, j) = (0, \dots, 1, \dots 0) \cdot (a_{1j}, \dots a_{nj}) = a_{ij},$$

  because the 1 in the i-th row of E is in the i-th place.

- In the location (l, j), for any j, we know that the entry there is just the dot product of E's k-th row and A's j-th column: i.e.

  $$\text{entry } (k, j) = (0, \dots, 0, \lambda, 0, \dots, 0, 1, 0, \dots 0) \cdot (a_{1j}, \dots a_{nj}) = \lambda a_{kj} + a_{lj},$$

  because the λ in the l-th row of E is in the k-th place, and the 1 is in the l-th place.

By inspection, this matrix is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{l-1,1} & a_{l-1,2} & a_{l-1,3} & a_{l-1,4} & a_{l-1,5} & \dots & a_{l-1,n} \\ \lambda a_{k1} + a_{l1} & \lambda a_{k2} + a_{l2} & \lambda a_{k3} + a_{l3} & \lambda a_{k4} + a_{l4} & \lambda a_{k5} + a_{l5} & \dots & \lambda a_{kn} + a_{ln} \\ a_{l+1,1} & a_{l+1,2} & a_{l+1,3} & a_{l+1,4} & a_{l+1,5} & \dots & a_{l+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & a_{n5} & \dots & a_{nn} \end{bmatrix}.$$

This is A with λ times its k-th row added to its l-th row, as claimed. $\square$

2 Elementary Matrices: How To Undo Them

On our last HW, set 14, we talked about how to "undo" or "invert" certain matrices: i.e. given a matrix like

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

we looked for a second matrix like

$$B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

such that

$$B \circ A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} (1,0,0)\cdot(1,1,1) & (1,0,0)\cdot(0,1,1) & (1,0,0)\cdot(0,0,1) \\ (-1,1,0)\cdot(1,1,1) & (-1,1,0)\cdot(0,1,1) & (-1,1,0)\cdot(0,0,1) \\ (0,-1,1)\cdot(1,1,1) & (0,-1,1)\cdot(0,1,1) & (0,-1,1)\cdot(0,0,1) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ the identity matrix.}$$

In essence, we were looking for a matrix that we can multiply A by and get the identity map: in other words, a linear map B that we can apply to A to "invert" or "undo" what A does! (This is because if $B \circ A$ is the identity map, then we're effectively saying that $B \circ A(\vec{x}) = \vec{x}$, for any vector $\vec{x}$. In other words, the map B "undoes" whatever A does to the vector $\vec{x}$.)

For some matrices, we can do this, like in the example above! For others, we can't: to see a trivial example, consider a matrix A with one column that's all 0's.

$$\begin{bmatrix} a_{11} & \dots & a_{1,k-1} & 0 & a_{1,k+1} & \dots & a_{1n} \\ a_{21} & \dots & a_{2,k-1} & 0 & a_{2,k+1} & \dots & a_{2n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{n,k-1} & 0 & a_{n,k+1} & \dots & a_{nn} \end{bmatrix}.$$

If we multiply this by any matrix B, we'll get

$$B \circ A = \begin{bmatrix} b_{11} & \dots & b_{1,k-1} & b_{1,k} & b_{1,k+1} & \dots & b_{1n} \\ b_{21} & \dots & b_{2,k-1} & b_{2,k} & b_{2,k+1} & \dots & b_{2n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{n,k-1} & b_{n,k} & b_{n,k+1} & \dots & b_{nn} \end{bmatrix} \circ \begin{bmatrix} a_{11} & \dots & a_{1,k-1} & 0 & a_{1,k+1} & \dots & a_{1n} \\ a_{21} & \dots & a_{2,k-1} & 0 & a_{2,k+1} & \dots & a_{2n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{n,k-1} & 0 & a_{n,k+1} & \dots & a_{nn} \end{bmatrix}$$

The matrix $B \circ A$ will have its k-th column made up entirely of 0's, as the entries in that column are all of the form (some row of B)$\cdot$(the k-th column of A, which contains only 0's). In particular, this means it cannot be the identity matrix, because the identity matrix's k-th column does not contain only 0's in it!

So: can we invert our elementary matrices?

Theorem. Let E be an elementary matrix. Then,

- if $E = E_{\text{multiply entry k by } \lambda}$, then E is invertible whenever $\lambda \neq 0$, and has inverse $E_{\text{multiply entry k by } (1/\lambda)}$.
- if $E = E_{\text{switch entry } k \text{ and entry } l}$, then E is invertible, and has inverse equal to itself.
- if $E = E_{\text{add } \lambda \text{ copies of entry } k \text{ to entry } l}$, then E is invertible, and has inverse $E_{\text{add } -\lambda \text{ copies of entry } k \text{ to entry } l}$.

Proof. These are pretty immediate, and follow from the properties of these matrices that we've proven earlier. For example, if $E = E_{\text{multiply entry k by } \lambda}$, then E is the matrix with 1's on its diagonal and 0's elsewhere, except for a λ at (k, k). If we multiply this on the left by $E_{\text{multiply entry k by } (1/\lambda)}$, then (by our earlier work) we're multiplying the k-th row of our original E by $(1/\lambda)$. This gives us the identity matrix.

The other two are similar. If $E = E_{\text{switch entry } k \text{ and entry } l}$, then E is the identity matrix with its k-th and l-th rows swapped. Multiplying on the left by E just swaps them again, which yields the identity matrix.

Finally, if $E = E_{\text{add } \lambda \text{ copies of entry } k \text{ to entry } l}$, then E is the identity matrix except for a λ in row l, column k. Applying $E_{\text{add } -\lambda \text{ copies of entry } k \text{ to entry } l}$ to this matrix adds $-\lambda$ copies of the k-th row of E to its l-th row. But this just adds $-\lambda$ copies of $\vec{e_k}$ to the l-th row, which is just adding $-\lambda$ to the cell that contains λ and doing nothing else. This yields the identity matrix, as claimed. □

3 Elementary Matrices: What Can We Make?

Here's a fun[1] game: suppose you're stranded on a desert island, with only a box containing all of the 3×3 elementary matrices to keep you company.

If you're allowed to compose these elementary matrices with each other, what other kinds of matrices can you make?

[1] For certain values of fun.

Well: you can certainly make something like

$$\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\circ\begin{bmatrix}1&0&0\\0&1&0\\4&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\4&0&1\end{bmatrix}.$$

And even something like

$$\begin{aligned}\left(\begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}\circ\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\right)\circ\begin{bmatrix}1&0&0\\0&1&0\\4&0&1\end{bmatrix}&=\begin{bmatrix}1&0&0\\2&1&0\\4&0&1\end{bmatrix}\circ\begin{bmatrix}1&0&0\\0&1&0\\4&0&1\end{bmatrix}\\&=\circ\begin{bmatrix}1&0&0\\2&1&0\\8&0&1\end{bmatrix}.\end{aligned}$$

And maybe even

$$\begin{aligned}\left(\begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}\circ\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\circ\begin{bmatrix}1&0&0\\0&1&0\\4&0&1\end{bmatrix}\right)\circ\begin{bmatrix}1&0&0\\0&1&0\\0&0&-3\end{bmatrix}&=\begin{bmatrix}1&0&0\\2&1&0\\8&0&1\end{bmatrix}\circ\begin{bmatrix}1&0&0\\0&1&0\\0&0&-3\end{bmatrix}\\&=\begin{bmatrix}1&0&0\\2&1&0\\-24&0&-3\end{bmatrix}.\end{aligned}$$

But what can't we create? Is there anything we can't create?