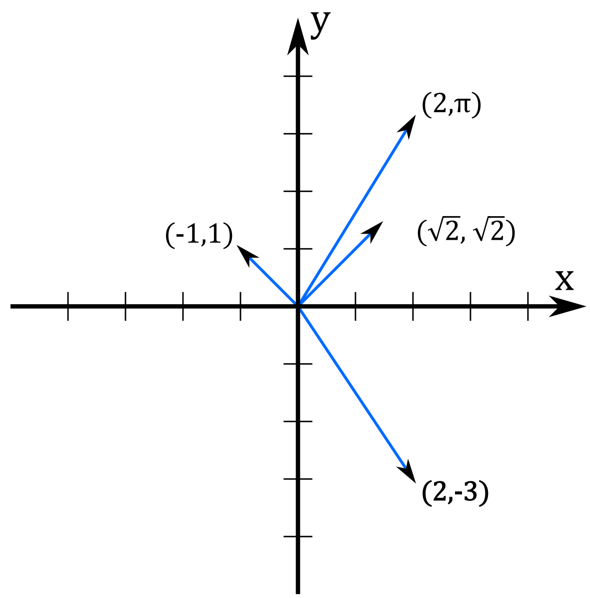
Math/CS 103	Professor: Padraic Bartlett
Lecture 2: Vector Spaces	
Week 1	*UCSB 2013*

Hello! This mini-lecture is designed to introduce the concepts of **vector spaces** and **span**, a pair of concepts we studied on the past homework.

1 Vector Spaces: Examples

As often happens in mathematics, the concept of a **vector space** is one that people usually understand examples of long before they run into their proper *definition*. Most students run into the vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ in their early geometry/precalc classes, which (due to their remarkably visual presentations) are often easier to deal with than the abstract definition of a vector space itself. For completeness' sake, we redefine these two vector spaces here:

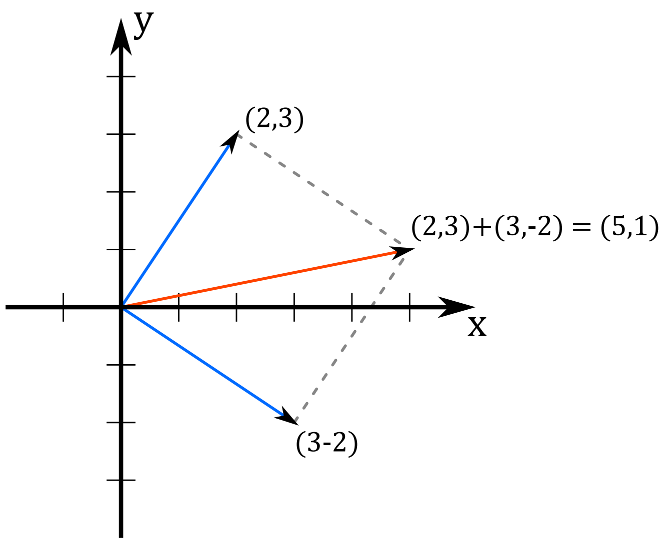
Definition. The **vector space** $\mathbb{R}^2$ consists of the collection of all pairs (a, b), where a, b are allowed to be any pair of real numbers. For example, $(2, -3), (2, \pi), (-1, 1)$, and $(\sqrt{2}, \sqrt{2})$ are all examples of vectors in $\mathbb{R}^2$. We typically visualize these vectors as arrows in the xy-plane, with the tail of the arrow starting at the origin[1] and the tip of the arrow drawn at the point in the plane with xy-coördinates given by the vector. We draw four such vectors here:



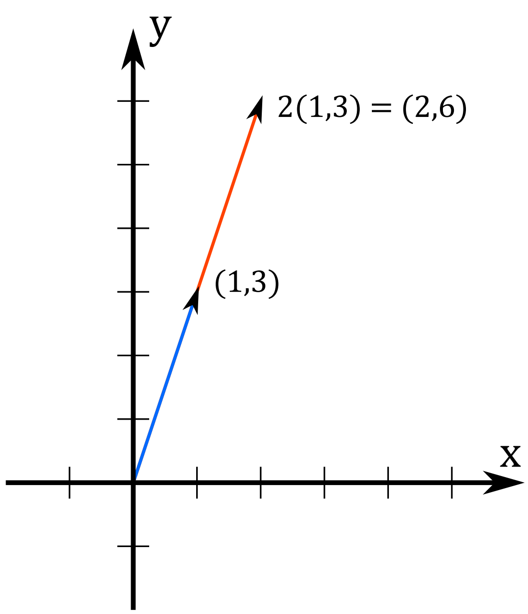
Given a pair of vectors in $\mathbb{R}^2$, we can **add** them together. We do this component-wise, i.e. if we have two vectors (a, b) and (c, d), their sum is the vector $(a+c, b+d)$. For example, the sum of the vectors $(3, -2)$ and $(2, 3)$ is the vector $(5, 1)$.

[1] The origin is the point $(0, 0)$ in the plane.

You can visualize this by taking the arrow corresponding to the first vector that we add, and "translating" this arrow over to the start of the second vector; if you travel along the first vector and then continue along this second translated vector, you arrive at some point in the plane. The arrow connecting the origin to this point is the vector given by the sum of these two vectors! If this seems hard to understand, the diagram below may help some:

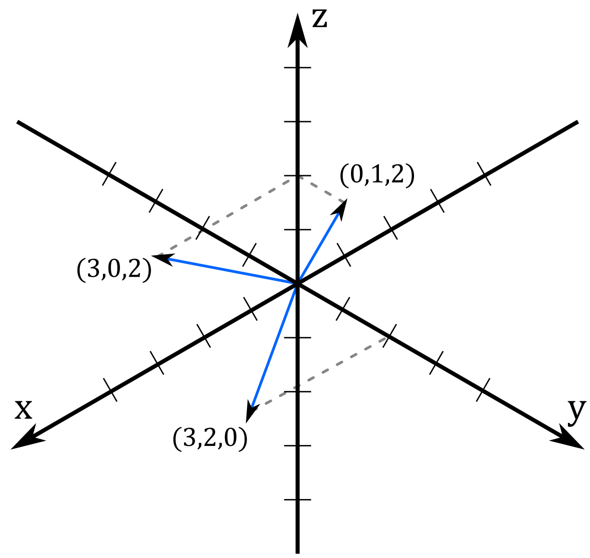


We can also **scale** a vector in $\mathbb{R}^2$ by any real number a. Intuitively, this corresponds to the concept of "stretching:" the vector (x, y) scaled by a, denoted $a(x, y)$, is the quantity (ax, ay). For example, $2(1, 3) = (2, 6)$, and is essentially what happens if we "double" the vector $(1, 3)$. We illustrate this below:

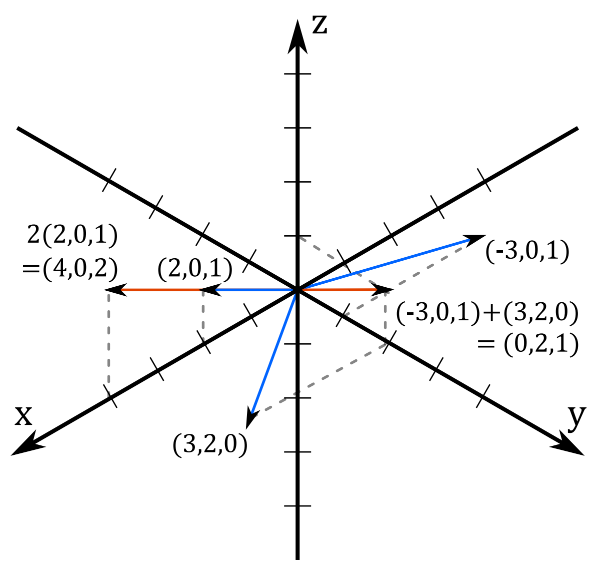


We can define $\mathbb{R}^3$ in a similar fashion:

Definition. The **vector space** $\mathbb{R}^3$ consists of the collection of all pairs (a, b, c), where a, b, c are allowed to be any triple of real numbers. For example, $(0, 1, 2), (3, 0, 2)$, and $(3, 2, 0)$ are all examples of vectors in $\mathbb{R}^3$. We typically visualize these vectors as arrows in three-dimensional xyz-space, with the tail of the arrow starting at the origin and the tip of the arrow drawn at the point in the plane with xyz-coördinates given by the vector. We draw three such vectors here:



Again, given a pair of vectors in $\mathbb{R}^3$, we can **add** them together. We do this component-wise, i.e. if we have two vectors (a, b, c) and (d, e, f), their sum is the vector $(a+d, b+e, c+f)$. For example, the sum of the vectors $(3, -2, 0)$ and $(2, 1, 2)$ is the vector $(5, -1, 2)$. We can also **scale** a vector in $\mathbb{R}^3$ by any real number a: the vector (x, y, z) scaled by a, denoted $a(x, y, z)$, is the quantity (ax, ay, az). These operations can be visualized in a similar fashion to the pictures we drew for $\mathbb{R}^2$:



You can generalize these examples to $\mathbb{R}^n$, the vector space made out of n-tuples of real numbers: i.e. elements of $\mathbb{R}^4$ would be things like $(\pi, 2, 2, 1)$ or $(-1, 2, 1, -1)$. In general, there are many other kinds of vector spaces — essentially, anything with the two operations "addition" and "scaling" is a vector space (provided that those operations are well-behaved in certain specific ways.) For now, though, let's work with just $\mathbb{R}^n$, and specifically $\mathbb{R}^2$ and $\mathbb{R}^3$: these are the two vector spaces that we have the easiest time with visualizing, and will allow us to begin to grapple with most of the concepts we want to focus on in the next few weeks.

2 Vector Space Concepts: Span

Using the language of vector spaces, it's not too hard to see that the problems on the last homework set were all mostly variations on the following theme:

> "Consider some set of vectors $\{\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}\}$ in $\mathbb{R}^3$, along with some point $\vec{x} \in \mathbb{R}^3$. Can you scale and add these $\vec{A_i}$-vectors (given some set of restrictions) together in a way to get $\vec{x}$?"

This class of questions is a frequently-occurring one in mathematics, so much so that mathematicians have invented a concept, called **span**, to describe this idea:

Definition. Take some collection of vectors $\{\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}\}$, all from some vector space[2] $\mathbb{R}^n$. Call this collection C. The **span** of C is the collection of all vectors that we can create by scaling and combining elements of C. In other words,

$$\text{span}(C) = \{\vec{w} \in \mathbb{R}^n | \text{ there are constants } a_1, \ldots a_n \in \mathbb{R} \text{ such that } \vec{w} = a_1\vec{v_1} + a_2\vec{v_2} + \ldots + a_n\vec{v_n}\}.$$

Relatedly, we call elements of the form $a_1\vec{v_1} + a_2\vec{v_2} + \ldots + a_n\vec{v_n}$ **linear combinations** of the elements of C.

In this sense, the problems on the first homework set were asking you whether certain points lay in the span (or restricted notions of span) of various sets of vectors. We provide a pair of sample calculations of spans, to help them make sense formally:

Question. Consider the collection $S = \{(1, 1, 1), (0, 1, 1), (1, 1, 0)\}$. Is the vector $(3, 4, 5)$ in the span of this collection?

Answer. Notice that because $(1, 1, 1)$ and $(0, 1, 1)$ are in our collection S, their difference

$$(1, 1, 1) - (0, 1, 1) = (1, 0, 0)$$

is also in our set. Similarly, for any $x \in \mathbb{R}$, we can generate the element $(x, 0, 0)$ via the following linear combination:

$$x(1, 1, 1) - x(0, 1, 1) = (x, 0, 0)$$

[2]This definition still holds for other vector spaces than $\mathbb{R}^n$, and we will revisit it when we get to some other examples of vector spaces.

In general, if an element $\vec{v}$ is in the span of some collection of vectors, any scalar multiple of $\vec{v}$ is also in this span, because we can just scale all of the elements used to make $\vec{v}$! As well, if two elements $\vec{v}, \vec{w}$ are in the span of a set, their sum is in the span of the set, because we can just sum all of the elements used to create $\vec{v}$ and $\vec{w}$ together. (If you don't see why these statements are true, prove them!)

Notice as well that the elements

$$(1, 1, 1) - (1, 1, 0) = (0, 0, 1)$$

and

$$(1, 1, 0) + (0, 1, 1) - (1, 1, 1) = (0, 1, 0)$$

are also in our span; consequently, the elements $(0, y, 0)$ and $(0, 0, z)$ are in our set, for any $y, z \in \mathbb{R}$. In summary, we've shown that for any $x, y, z \in \mathbb{R}$, we have $(x, 0, 0), (0, y, 0)$, and $(0, 0, z)$ in the span of S. Therefore, by summing these three elements together, we've actually shown that for any $x, y, z \in \mathbb{R}$, (x, y, z) is in the span of S. In other words, the span of S is all of $\mathbb{R}^3$, and therefore it contains all vectors in $\mathbb{R}^3$.

In particular, it contains $(3, 4, 5)$.

Question. Consider the collection $S = \{(9, 8, 7), (6, 5, 4), (3, 2, 1)\}$. Is the vector $(6, 7, 8)$ in the span of this collection?

Answer. First, notice that the difference between $(9, 8, 7)$ and $(6, 5, 4)$ is $(3, 3, 3)$, so $(3, 3, 3)$ is in our set. Moreover, we can use this observation to see that because

$$(6, 5, 4) - (3, 3, 3) = (3, 2, 1),$$

that we can create the element $(3, 2, 1)$ using only multiples of $(9, 8, 7)$ and $(6, 5, 4)$. (In specific, we just showed that it was equal to $2(6, 5, 4) - (9, 8, 7)$.)

Therefore, the element $(3, 2, 1)$ is in some sense "irrelevant" to the span of our collection: if we can recreate it with the elements $(9, 8, 7)$ and $(6, 5, 4)$, there's really no need to consider it in addition to these two elements. In other words, the span of just $(9, 8, 7)$ and $(6, 5, 4)$ on their own is the same as the span of $\{(9, 8, 7), (6, 5, 4), (3, 2, 1)\}$! (Again, if this is unclear, prove this claim.)

This simplifies things: we know that the span of our set S is just the collection of all vectors of the form

$$(9a + 6b, 8a + 5b, 7a + 4b),$$

for $a, b \in \mathbb{R}$. We want to know if $(6, 7, 8)$ is in this collection. If it is, then there must be some a, b such that

$$(9a + 6b, 8a + 5b, 7a + 4b) = (6, 7, 8);$$

i.e. some a, b such that

$$\begin{aligned} 9a + 6b &= 6 \\ 8a + 5b &= 7 \\ 7a + 4b &= 8. \end{aligned}$$

Our first equation,

$$9a + 6b = 6,$$

becomes

$$b = 1 - \frac{3}{2}a$$

after solving for b in terms of a.

If we plug this into our third equation, we get

$$7a + 4\left(1 - \frac{3}{2}a\right) = 8,$$

which simplifies to

$$7a + 4 - 6a = 8,$$

i.e. $a = 4$. Plugging $a = 4$ into this third equation again, but leaving the b in this time, gives us

$$7 \cdot 4 + 4b = 8,$$

which forces $b = -5$.

Plugging $a = 4, b = -5$ in each of the above equations works. Consequently, if we put these three equations back together into one vector, we've just shown that

$$4(9, 8, 7) - 5(6, 5, 4) = (6, 7, 8).$$

Therefore, $(6, 7, 8)$ is in the span of our set.