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Lecture 4: Basis and Dimension

Week 1 *UCSB 2013*

Hello! This mini-lecture is designed to introduce the concepts of **basis** and **dimension**. We do this below:

1 Basis and Dimension

On your third problem set, you dealt with a number of questions that asked you if you could create submarines capable of visiting any point in the ocean (usually given some set of constraints.) If we think of the ocean as simply $\mathbb{R}^3$, and as our submarines as just collections of vectors (corresponding to their engines,) these questions can be abstracted to the following kind of problem:

> "Given the constraint *(blah)*, can we find a collection of vectors that satisfy this constraint that span $\mathbb{R}^3$?"

A related question to this was the idea of minimality, which we also studied on this problem set. If you remove the submarine framing, question 4 essentially asked the following:

> "What is the smallest number of vectors you need to span $\mathbb{R}^3$?"

Variations on these two questions come up often in linear algebra! Consequently, we came up with the following terms to work with these ideas:

Definition. Let S be some collection of vectors in $\mathbb{R}^m$. We say that this collection of vectors is a **basis** for $\mathbb{R}^m$ if the following properties hold:

1. The span of S is all of $\mathbb{R}^m$.
2. The collection S is linearly independent.

In this sense, we're asking that S both spans the entire space that we're considering, and also that no vector in S is "superfluous."

This is because if S **was** a linearly dependent set, then there would be some set of vectors $\vec{v_1}, \ldots \vec{v_n} \in S$ and coefficients $a_1, \ldots a_n$ not all zero such that

$$a_1\vec{v_1} + a_2\vec{v_2} + \ldots + a_n\vec{v_n} = \vec{0}.$$

Consequently, we could solve this equation for $\vec{v_1}$ and get

$$\vec{v_1} = -\frac{a_2}{a_1}\vec{v_2} - \frac{a_3}{a_1}\vec{v_2} - \ldots \frac{a_n}{a_1}\vec{v_n} := \star.$$

This is a way to create the vector $\vec{v_1}$ without using the vector $\vec{v_1}$ itself! Therefore, we know that the span of S is the same thing as the span of S without the vector $\vec{v_1}$: this is because anything we can make with linear combinations of elements of S, we can do without $\vec{v_1}$, by just replacing every copy of $\vec{v_1}$ with $\star$. In other words, the vector $\vec{v_1}$ is superfluous: we can do without it!

Definition. The **dimension** of $\mathbb{R}^n$, or indeed any vector space, is the number of elements needed to create a basis for the space.

We study a quick example to illustrate the ideas here:

Question. What is the dimension of $\mathbb{R}^n$? Can you find a basis for $\mathbb{R}^n$? Can you find a basis for, say, $\mathbb{R}^2$ and $\mathbb{R}^3$ made out of vectors that have nonzero x, y and z coördinates?

Answer. For the moment, let's look at $\mathbb{R}^2$. We claim that the pair of vectors $(1,0)$ and $(0,1)$ form a basis for $\mathbb{R}^2$. To see this, simply notice that we can express any vector in $\mathbb{R}^2$ as $x(1,0)+y(0,1)$, so it spans the entire space. Furthermore, if we have $x(1,0)+y(0,1) = (0,0)$, we would have $x = 0 = y$. Therefore, this pair of vectors is linearly independent as well.

In general: for $\mathbb{R}^n$, the vectors $(1,0,0,\ldots 0), (0,1,0,0\ldots 0), \ldots (0,0,\ldots 0,1)$ form a basis for $\mathbb{R}^n$, for the same reasons: there is no combination of these vectors that sums to 0 without having all of the coefficients equal to 0, and we can create any vector $(a_1, \ldots a_n)$ by summing $a_1(1,0\ldots 0) + \ldots a_n(0,\ldots 0,1)$.

Therefore, the dimension of $\mathbb{R}^n$ is n, which is reassuring.

We now consider the second part of our question: can we make a basis for $\mathbb{R}^n$ where none of the components of the vectors in our basis are 0?

Again, we start with $\mathbb{R}^2$. Here, we can do this: try $(1,1)$ and $(1,-1)$. We want to combine these two vectors to get any element of the form (x,y). To do this, we simply solve the equations $a(1,1) + b(1,-1) = (x,y)$ for a, b:

$$\begin{aligned}
a + b &= x \\
a - b &= y \\
\Rightarrow 2a &= x + y, \text{ by adding these two equations.} \\
\Rightarrow a &= \frac{x+y}{2}. \\
\Rightarrow \frac{x+y}{2} + b &= x, \text{ by substituting in for } a \text{ in the first equation.} \\
\Rightarrow b &= \frac{x-y}{2}.
\end{aligned}$$

Therefore, we can create any vector with this pair, via the linear combination

$$\frac{x+y}{2}(1,1) + \frac{x-y}{2}(1,1) = (x,y).$$

Furthermore, we know that this is a basis. To see this, notice that if we had $(x,y) = (0,0)$, our earlier work has just shown that $a = \frac{x+y}{2} = 0$ and $b = \frac{x-y}{2} = 0$. Therefore, the only linear combination of our two vectors that yields $(0,0)$ has identically zero coefficients, which is the definition of linear independence.

(There are lots of solutions here! This was just a pair of vectors I wanted to try because they looked interesting. The same holds for our next example in $\mathbb{R}^3$: there are many solutions, and I have picked these basically at random.)

For $\mathbb{R}^3$, we try $(1,1,1),(1,-1,-1),(1,1,-1)$. Again, we want to combine these two vectors to get any element of the form (x,y,z). To do this, we simply solve the equations $a(1,1,1)+b(1,-1,-1)+c(1,1,-1)=(x,y,z)$ for a,b,c:

$$\begin{aligned}
a+b+c&=x\\
a-b+c&=y\\
a-b-c&=z\\
\Rightarrow 2a&=x+z,\ \text{by adding the first and third equations.}\\
\Rightarrow a&=\frac{x+z}{2}.\\
\text{Also, } 2a+2c&=x+y,\ \text{by adding the first two equations.}\\
\Rightarrow x+z+2c&=x+y.\\
\Rightarrow c&=\frac{y-z}{2}.\\
\Rightarrow \frac{x+z}{2}+b+\frac{y-z}{2}&=x,\ \text{by substituting } a,b \text{ in the first equation.}\\
\Rightarrow b&=\frac{x-y}{2}.
\end{aligned}$$

Therefore, we can create any vector with this triple, via the linear combination

$$\frac{x+z}{2}(1,1,1)+\frac{x-y}{2}(1,-1,1)+\frac{y-z}{2}(1,-1,-1)=(x,y,z).$$

Furthermore, we know that this is a basis. To see this, notice that if we had $(x,y,z)=(0,0,0)$, our earlier work has just shown that $a=\frac{x+z}{2}=0$, $b=\frac{x-y}{2}=0$ and $c=\frac{y-z}{2}=0$. Therefore, the only linear combination of our three vectors that yields $(0,0,0)$ has identically zero coefficients, which is the definition of linear independence.