| Math/CS 103 | Professor: Padraic Bartlett |
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# Lecture 5: Polynomials and Vector Spaces

| *Week 2* | *UCSB 2013* |
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Hello! This mini-lecture is designed to introduce the concept of **polynomials** in the language of vector spaces. We do this below:

## 1 Polynomials and Vectors

Most of our problem sets thus far have been about $\mathbb{R}^n$, and studying the properties of this object as a **vector space**. We've done so in a number of different ways, introducing terms like **span**, **linear dependence and independence**, **basis**, and **dimension** to study this object.

However: the only properties we've been using thus far about $\mathbb{R}^n$, in a sense, is that we have two operations on it:

- Vector addition: given two vectors, we can add them together.
- Scalar multiplication: given a vector, we can multiply it by a real number.

We know that these two operations are well-behaved: i.e. that scalar-multiplying anything by 1 doesn't change it, that $(0, 0 \ldots 0)$ added to any vector never changes it, and lots of common-sense properties.

This, however, raises a natural question: can we study **anything** that has a nice addition and scalar multiplication property with the vector space language we've been developing?

Specifically: on your fourth problem set, you started to study **polynomials**. Polynomials are things that we can add and multiply by real numbers. So can we treat these like vector spaces?

The answer is yes! And in fact, this is what we did on this problem set. Question 1 asked you to pick out of a set of five polynomials a subset that can be combined to create any degree-3 polynomial. In other words, you were asked to pick out a set that **spanned** the space of degree-$\leq 3$ polynomials!

You can even look at this like they're elements in $\mathbb{R}^n$: a degree-3 polynomial $a + bx + cx^2 + dx^3$ can be thought of as just a 4-tuple $(a, b, c, d)$. In this sense, you were just finding a set that would span all of $\mathbb{R}^4$!

The other questions were similar: in problem 2, you were studying whether your subset of chips was linearly independent, or dependent: i.e. whether you could remove a chip and still generate every element in your space. Problems 3 and 4 asked you to consider certain subsets and decide whether a basis for the space of all degree $\leq 3$ can be created using those subsets; problem 5 was asking you to show that the dimension of all degree-$\leq 3$ polynomials is 4.

So! Define the set of all polynomials with finite degree and real-valued coefficients to be the set $\mathbb{R}[x]$. We'll think of this as a vector space in this class, as you've seen in the homework. To further illustrate this idea that this is indeed a vector space, we work an example:

**Question.** Consider the set of polynomials $\{1 + 2x + 3x^2, 2 + 3x + 4x^2, 3 + 4x + 5x^2\}$ as vectors in the vector space $\mathbb{R}[x]$. Show that this set is linearly dependent, and find a linearly independent subset of this set.

**Answer.** To see that this set is linearly dependent, you can either think back to HW#3, p1, part (c) (where you proved this) or follow the following proof: consider the difference

$$(2 + 3x + 4x^2) - (1 + 2x + 3x^2) = 1 + x + x^2.$$

We can add a copy of this to $2 + 3x + 4x^2$ to get the third element in our set, $3 + 4x + 5x^2$. Therefore, we've just shown that

$$(2 + 3x + 4x^2) + ((2 + 3x + 4x^2) - (1 + 2x + 3x^2)) = 3 + 4x + 5x^2;$$

in other words, that we have created a nontrivial linear combination of the first two elements in our set that is equal to the third one! If we subtract $(3 + 4x + 5x^2)$ from both sides, we get a nontrivial linear combination of elements in our set that is 0: in other words, we've shown that our set is linearly dependent.

However, we also can see that $\{1 + 2x + 3x^2, 2 + 3x + 4x^2\}$ is linearly independent. To see why, take any linear combination of these two polynomials that sums to 0: i.e. any $a, b$ such that

$$a(1 + 2x + 3x^2) + b(2 + 3x + 4x^2) = 0.$$

By summing like powers of $x$, we have that $a + 2b = 0$ and $2a + 3b = 0$; in other words, $a = -2b$ and $a = -\frac{3}{2}b$. The only solution to these two equations is $a = b = 0$. Therefore, there is no nontrivial combination of our two polynomials that sums to 0: i.e. our set is linearly independent!