Math/CS 103	Professor: Padraic Bartlett
Lecture 5: Dot Products	
Week 2	*UCSB 2013*

Hello! This mini-lecture is designed to introduce the **dot product**. We do this below:

1 Dot Products

Homework 5 is basically centered around the **dot product** operation, defined here:

Definition. Take two vectors $(x_1, \ldots x_n), (y_1, \ldots y_n) \in \mathbb{R}^n$. Their **dot product** is simply the sum

$$x_1y_1 + x_2y_2 + \ldots x_ny_n.$$

In homework 5, we asked whether we could find sets of vectors or sequences that had various properties with respect to the dot product: for example, we asked for sets of vectors that had pairwise 0 dot product, pairwise negative dot product, "small" dot product, etc.

In this talk, we instead focus on a geometric interpretation of the dot product, as given by the following theorem. Before stating it, we make the following definition: given a vector $\vec{x}$, we let $||\vec{x}||$ denote the length of this vector. If this vector $\vec{x}$ is a vector in $\mathbb{R}^3$ of the form (a, b, c), this length is simply the quantity $\sqrt{a^2 + b^2 + c^2}$, which you can see by using the Pythagorean theorem. (If you have questions on this part, contact me!)

With this stated, we can move on to our theorem:

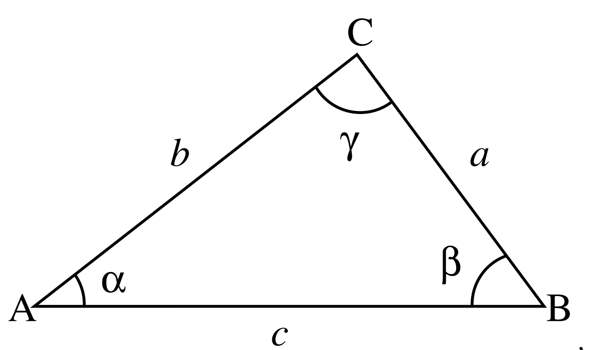
Theorem. Let $\vec{x} = (x_1, x_2, x_3), \vec{y} = (y_1, y_2, y_3)$ be a pair of vectors in $\mathbb{R}^3$. Then $\vec{x} \cdot \vec{y}$ is equal to

$$||\vec{x}|| \cdot ||\vec{y}|| \cos(\theta),$$

where θ is the angle between $\vec{x}$ and $\vec{y}$.

Proof. Essentially, this is a consequence of the Law of Cosines, a trig rule you may have ran into in high school. We restate it here:

Proposition. (Law of Cosines) Given the triangle

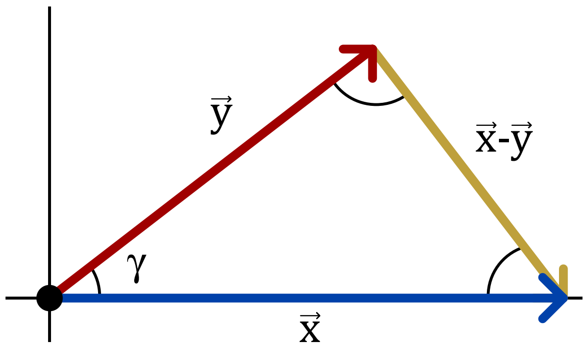


,

we have the equality

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma).$$

To see how this applies to our situation, consider the following picture:



If we apply the law of cosines to this image, we get

$$||\vec{x} - \vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2 - 2||\vec{x}||||\vec{y}||\cos(\gamma).$$

However, we know that the length of $\vec{x} - \vec{y}$ is just the length of the vector $(x_1 - y_1, x_2 - y_2, x_3 - y_3)$, which is

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y^2)^2 + (x_3 - y_3)^2}.$$

Therefore, if we square this, we get

$$\begin{aligned}
||\vec{x} - \vec{y}||^2 &= (x_1^2 - 2x_1y_1 + y_1^2) + (x_2^2 - 2x_2y_2 + y_2^2) + (x_3^2 - 2x_3y_3 + y_3^2) \\
&= (x_1^2 + x_2^2 + x_3^2) + (y_1^2 + y_2^2 + y_3^2) - 2(x_1y_2 + x_2y_2 + x_3y_3) \\
&= ||\vec{x}||^2 + ||\vec{y}||^2 - 2\vec{x} \cdot \vec{y}.
\end{aligned}$$

If we plug this into our law of cosines formula, we get

$$\begin{aligned}
||\vec{x}||^2 + ||\vec{y}||^2 - 2\vec{x} \cdot \vec{y} &= ||\vec{x}||^2 + ||\vec{y}||^2 - 2||\vec{x}||||\vec{y}||\cos(\gamma) \\
\Rightarrow \vec{x} \cdot \vec{y} &= ||\vec{x}||||\vec{y}||\cos(\gamma).
\end{aligned}$$

So we've proven our claim! □